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Primitive decompositions of Johnson graphs ☆

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Abstract

A *transitive decomposition* of a graph is a partition of the edge set together with a group of automorphisms which transitively permutes the parts. In this paper we determine all transitive decompositions of the Johnson graphs such that the group preserving the partition is arc-transitive and acts primitively on the parts.

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1. Introduction

A *decomposition* of a graph is a partition of the edge set with at least two parts, which we interpret as subgraphs and call the *divisors* of the decomposition. If each divisor is a spanning subgraph we call the decomposition a *factorisation* and the divisors *factors*. Graph decompositions and factorisations have received much attention, see for example [2,23]. Of particular

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interest [21,22] are decompositions where the divisors are pairwise isomorphic. These are known as *isomorphic decompositions*.

A *transitive decomposition* is a decomposition $\mathcal{P}$ of a graph Γ together with a group of automorphisms G which preserves the partition and acts transitively on the set of divisors. We refer to $(\Gamma, \mathcal{P})$ as a G-transitive decomposition. This is a special class of isomorphic decompositions and a general theory has been outlined in [20]. Sibley [33] has described all G-transitive decompositions of the complete graph K_n where G is 2-transitive on vertices. This generalised the Cameron–Korchmáros classification in [7] of the G-transitive 1-factorisations of K_n (that is, the factors have valency 1) with G acting 2-transitively on vertices. Note that a subgroup of S_n is arc-transitive on K_n if and only if it is 2-transitive. Also all G-transitive decompositions of graphs with G inducing a rank three product action on vertices have been determined in [1]. A special class of transitive decompositions, called *homogeneous factorisations*, are the G-transitive decompositions $(\Gamma, \mathcal{P})$ such that the kernel M of the action of G on $\mathcal{P}$ is vertex-transitive. This implies that each divisor is a spanning subgraph and so $\mathcal{P}$ is indeed a factorisation. Homogeneous factorisations were first introduced in [27] for complete graphs and extended to arbitrary graphs and digraphs in [19].

The *Johnson graph* $J(n,k)$ is the graph with vertices the k-element subsets of an n-set X, two sets being adjacent if they have $k-1$ points in common. Note that $J(n,1) \cong K_n$ and $J(n,k) \cong J(n,n-k)$ so we always assume that $2 \leqslant k \leqslant \frac{n}{2}$. Note that $J(4,2) \cong K_{2,2,2}$ while the complement of $J(5,2)$ is the Petersen graph. All homogeneous factorisations of $J(n,k)$ were determined in [11,12]. Examples only exist for $J(q+1,2)$ for prime powers $q \equiv 1 \pmod 4$, $J(q,2)$ and $J(q+1,3)$ for $q = 2^{r^f}$ with r an odd prime, and for $J(8,3)$. However, examples of transitive decompositions exist for all values of n and k (see Construction 2.8). Constructions 2.8(1) and (2) were drawn to our attention by Michael Orrison. Both constructions were used in [25] to help determine maximal subgroups of symmetric groups while Construction 2.8(1) was used in [30] for the statistical analysis of unranked data.

In this paper we determine all G-transitive decompositions of the Johnson graphs subject to two conditions on G. The first is that G is arc-transitive while the second is that G acts primitively on the set of divisors of the decomposition. We call G-transitive decompositions for which G acts primitively on the set of divisors, *G-primitive decompositions.* We see in Lemma 2.2 that any G-transitive decomposition is the refinement of some G-primitive decomposition. By Theorem 3.4, a subgroup $G \leqslant S_n$ acts transitively on the set of arcs of $J(n,k)$ if and only if G is $(k+1)$-transitive, or $(n,k) = (9,3)$ and $G = \mathrm{P\Gamma L}(2,8)$. Using this, we analyse the appropriate groups to determine all primitive decompositions. In particular we prove the following theorem.

Theorem 1.1. *Let G be an arc-transitive group of automorphisms of $\Gamma = J(n,k)$ where $2 \leqslant k \leqslant n/2$. If $(\Gamma, \mathcal{P})$ is a G-primitive decomposition then one of the following holds*:

(1) *the divisors are matchings or unions of cycles,*
(2) *the divisors are unions of K_{n-k+1}, K_{k+1} or K_3, or*
(3) *$(\Gamma, \mathcal{P})$ is given by one of the rows of Table* 1.

The divisor graphs Σ and Π of Table 1 are investigated further in [13]. Construction 2.10 allows us to construct transitive decompositions of $J(n,k)$ with divisors isomorphic to $J(l,k)$ for any Steiner system $S(k+1,l,n)$ and this accounts for many of the examples in Table 1. Further constructions of transitive decompositions from Steiner systems are given in Section 2 and these have divisors isomorphic to unions of cliques or matchings.

Table 1
G-primitive decompositions of $J(n,k)$ for Theorem 1.1

Γ	G	Divisor	Comments
$J(6,3)$	A_6 or $\langle A_6, (1,2)\tau\rangle$	Petersen graph	Example 4.3(2)
$J(12,4)$	M_{12}	$2J(6,4)$	Constructions 2.10 and 2.1
$J(12,4)$	M_{12}	Σ	Construction 5.6
$J(24,4)$	M_{24}	$J(8,4)$	Construction 2.10
$J(23,3)$	M_{23}	$J(7,3)$	Construction 2.10
$J(11,3)$	M_{11}	$J(5,2)$	Construction 2.10
$J(11,3)$	M_{11}	2 Petersen graphs	Construction 6.11
$J(11,3)$	M_{11}	11 Petersen graphs	Construction 6.10(2)
$J(11,3)$	M_{11}	Π	Construction 6.10(1)
$J(9,3)$	$\mathrm{P\Gamma L}(2,8)$	$\mathrm{PSL}(2,8)$-orbits	Construction 6.13(1)
$J(9,3)$	$\mathrm{P\Gamma L}(2,8)$	Heawood graph	Construction 6.13(4)
$J(22,2)$	M_{22} or $\mathrm{Aut}(M_{22})$	$J(6,2)$	Construction 2.10
$J(2^d,2)$, $d \geqslant 3$	$\mathrm{AGL}(d,2)$	$2^{d-2}K_{2,2,2}$	Constructions 2.10 and 2.1
$J(16,2)$	$C_2^4 \rtimes A_7$	$4K_{2,2,2}$	Constructions 2.10 and 2.1
$J(q+1,2)$	3-transitive subgroup of $\mathrm{P\Gamma L}(2,q)$	$J(q_0+1,2)$ $q=q_0^r$, r prime	Construction 2.10
$J(q+1,2)$ $q \equiv 1 \pmod 4$	3-transitive subgroup of $\mathrm{P\Gamma L}(2,q)$	$\mathrm{PSL}(2,q)$-orbits	Construction 8.1

2. General constructions

First we show that the study of transitive decompositions can be reduced to the study of primitive decompositions. We denote by $V\Gamma$, $E\Gamma$ and $A\Gamma$, the sets of vertices, edges and arcs respectively, of the graph Γ.

Construction 2.1. Let $(\Gamma,\mathcal{P})$ be a G-transitive decomposition and let $\mathcal{B}$ be a system of imprimitivity for G on $\mathcal{P}$. For each $B \in \mathcal{B}$, let $Q_B = \bigcup_{P\in B} P$ and let $\mathcal{Q} = \{Q_B \mid B \in \mathcal{B}\}$. Then $(\Gamma,\mathcal{Q})$ is a G-transitive decomposition.

Lemma 2.2. *Any G-transitive decomposition $(\Gamma,\mathcal{P})$ with $|\mathcal{P}|$ finite is the refinement of a G-primitive decomposition $(\Gamma,\mathcal{Q})$.*

Proof. If $G^{\mathcal{P}}$ is primitive then we are done. If not, let $\mathcal{B}$ be a nontrivial system of imprimitivity for G on $\mathcal{P}$ with maximal block size. Then $G^{\mathcal{B}}$ is primitive and $\mathcal{P}$ is a refinement of the partition $\mathcal{Q}$ yielded by Construction 2.1. Thus $(\Gamma,\mathcal{Q})$ is a G-primitive decomposition. $\square$

We have the following general construction of transitive decompositions.

Construction 2.3. Let Γ be a graph with an arc-transitive group G of automorphisms. Let e be an edge of Γ and suppose that there exists a subgroup H of G such that $G_e < H < G$. Let $P = e^H$ and $\mathcal{P} = \{P^g \mid g \in G\}$.

Lemma 2.4. *Let $(\Gamma,\mathcal{P})$ be obtained as in Construction* 2.3. *Then $(\Gamma,\mathcal{P})$ is a G-transitive decomposition. Conversely, every G-transitive decomposition with G arc-transitive arises in such a manner. Moreover, if the subgroup H is maximal in G, then $(\Gamma,\mathcal{P})$ is a G-primitive decomposition.*

Proof. Since G is arc-transitive and $G_e < H < G$, then $\mathcal{P}$ is a partition of $E\Gamma$ which is preserved by G and such that $G^{\mathcal{P}}$ is transitive. Thus $(\Gamma, \mathcal{P})$ is a G-transitive decomposition. Conversely, let $(\Gamma, \mathcal{P})$ be a G-transitive decomposition such that G is arc-transitive. Let e be an edge of Γ and P the divisor containing e. Since $\mathcal{P}$ is a system of imprimitivity for G on $E\Gamma$ it follows that for $H = G_P$ we have $G_e < H < G$ and $P = e^H$. Moreover, $\mathcal{P} = \{P^g \mid g \in G\}$ and so $(\Gamma, \mathcal{P})$ arises from Construction 2.3. The last statement follows from the fact that H is the stabiliser in G of the divisor P. □

Remark 2.5. Lemma 2.4 implies that there are two possible ways to determine all G-transitive decompositions such that the divisor stabilisers are in a given conjugacy class H^G of subgroups of G. One is to fix an edge e and run over all subgroups conjugate to H which contain the stabiliser of e. Note that different conjugates may give different partitions. The second is to run over all edges whose stabiliser is contained in H. Again, different edges may give different partitions.

We say that two decompositions $(\Gamma, \mathcal{P}_1)$ and $(\Gamma, \mathcal{P}_2)$ are *isomorphic* if there exists $g \in \mathrm{Aut}(\Gamma)$ such that $\mathcal{P}_1^g = \mathcal{P}_2$. If both are G-transitive decompositions, then they are *isomorphic G-transitive decompositions* if there is such an element $g \in N_{\mathrm{Aut}(\Gamma)}(G)$. The following lemma gives us a condition for determining when different conjugates give the same decomposition.

Lemma 2.6. *Let $(\Gamma, \mathcal{P}_1)$, $(\Gamma, \mathcal{P}_2)$ be two G-transitive decompositions with G arc-transitive.*

(1) *Let e be an edge of Γ and P_1, P_2 be the divisors of $\mathcal{P}_1$, $\mathcal{P}_2$ respectively that contain e. If there exists an automorphism $g \in N_{\mathrm{Aut}(\Gamma)}(G)$ fixing e such that $G_{P_1}^g = G_{P_2}$ then $(\Gamma, \mathcal{P}_1)$ and $(\Gamma, \mathcal{P}_2)$ are isomorphic.*
(2) *Let e_1, e_2 be two edges of Γ with divisors $P_1 = e_1^H$ and $P_2 = e_2^H$ of $\mathcal{P}_1$, $\mathcal{P}_2$ respectively. If there exists an automorphism $g \in N_{\mathrm{Aut}(\Gamma)}(G)$ mapping e_1 onto e_2 such that $H^g = H$ then $(\Gamma, \mathcal{P}_1)$ and $(\Gamma, \mathcal{P}_2)$ are isomorphic.*

Proof. (1) By Lemma 2.4, $P_1 = e^{G_{P_1}}$ and $P_2 = e^{G_{P_2}}$. Thus $P_2 = e^{g^{-1}G_{P_1}g} = e^{G_{P_1}g} = P_1^g$. Moreover, $\mathcal{P}_2 = P_2^G = (P_1^g)^G = (P_1^G)^g = \mathcal{P}_1^g$ and so $(\Gamma, \mathcal{P}_1)$ and $(\Gamma, \mathcal{P}_2)$ are isomorphic.
(2) We have $P_2 = e_2^H = (e_1^g)^H = (e_1^H)^g = P_1^g$. Hence we get the same conclusion. □

We also have the following useful lemma.

Lemma 2.7. *Let $(\Gamma, \mathcal{P})$ be a G-primitive decomposition, with H the stabiliser of a divisor P. If $L \leqslant G$ is such that $L \nleqslant H$, L is arc-transitive on Γ and $L \cap H$ is maximal in L, then $(\Gamma, \mathcal{P})$ is a L-primitive decomposition.*

Proof. Since L is arc-transitive and contained in G, it follows that L acts transitively on $\mathcal{P}$. Moreover, since $H \cap L$ is the stabiliser in L of divisor P, it follows that L acts primitively on $\mathcal{P}$. □

We now describe some general methods for constructing transitive decompositions of Johnson graphs.

Construction 2.8. Let X be an n-set.

(1) For each $(k-1)$-subset Y of X, let P_Y be the complete subgraph of $J(n,k)$ whose vertices are all the k-subsets containing Y. Then

$$\mathcal{P}_\cap = \{P_Y \mid Y \text{ is a } (k-1)\text{-subset of } X\}$$

is a decomposition of $J(n,k)$ with $\binom{n}{k-1}$ divisors, each isomorphic to K_{n-k+1}.

(2) For each $(k+1)$-subset W of X, let Q_W be the complete subgraph whose vertices are all the k-subsets contained in W. Then

$$\mathcal{P}_\cup = \{Q_W \mid W \text{ is a } (k+1)\text{-subset of } X\}$$

is a decomposition of $J(n,k)$ with $\binom{n}{k+1}$ divisors, each isomorphic to K_{k+1}.

(3) For each $\{a,b\} \subseteq X$, let

$$M_{\{a,b\}} = \{\{\{a\} \cup Y, \{b\} \cup Y\} \mid Y \text{ a } (k-1)\text{-subset of } X\backslash\{a,b\}\}.$$

Then

$$\mathcal{P}_\ominus = \{M_{\{a,b\}} \mid \{a,b\} \subseteq X\}$$

is a decomposition of $J(n,k)$ with $\binom{n}{2}$ divisors, each of which is a matching with $\binom{n-2}{k-1}$ edges.

Given two sets A and B we denote the *symmetric difference* of A and B by $A \ominus B$.

Lemma 2.9. *Let $G \leqslant S_n$ such that $\Gamma = J(n,k)$ is G-arc-transitive. Let A and B be two adjacent vertices of Γ. Then $(\Gamma, \mathcal{P}_\cap)$, $(\Gamma, \mathcal{P}_\cup)$, $(\Gamma, \mathcal{P}_\ominus)$ are G-transitive decompositions. Moreover, if $G_{A\cap B}$, $G_{A\cup B}$, or $G_{A\ominus B}$ respectively is maximal in G, then the decomposition is G-primitive.*

Proof. Since $P_Y^g = P_{Y^g}$, $Q_W^g = Q_{W^g}$ and $M_{\{a,b\}}^g = M_{\{a,b\}^g}$, it follows that G preserves $\mathcal{P}_\cap$, $\mathcal{P}_\cup$ and $\mathcal{P}_\ominus$. Since G is arc-transitive, all three decompositions are G-transitive. The divisor of $\mathcal{P}_\cap$, $\mathcal{P}_\cup$ or $P_\ominus$ containing $\{A,B\}$ is $P_{A\cap B}$, $Q_{A\cup B}$ or $M_{A\ominus B}$ respectively, and the stabiliser of this divisor is $G_{A\cap B}$, $G_{A\cup B}$, or $G_{A\ominus B}$ respectively. The last assertion follows. □

Another method for constructing transitive decompositions of $J(n,k)$ is to use Steiner systems with multiply transitive automorphism groups. A *Steiner system* $S(t,k,v) = (X, \mathcal{B})$ is a collection $\mathcal{B}$ of k-subsets (called *blocks*) of a v-set X such that each t-subset of X is contained in a unique block.

Construction 2.10. Let $\mathcal{D} = (X, \mathcal{B})$ be an $S(k+1,l,n)$ Steiner system with automorphism group G such that G is transitive on $\mathcal{B}$. For each $Y \in \mathcal{B}$, let P_Y be the subgraph of $J(n,k)$ whose vertices are the k-subsets in Y and let $\mathcal{P} = \{P_Y \mid Y \in \mathcal{B}\}$.

Lemma 2.11. *The pair $(J(n,k), \mathcal{P})$ yielded by Construction* 2.10 *is a G-transitive decomposition with divisors isomorphic to $J(l,k)$. Moreover, the decomposition is G-primitive if and only if the stabiliser of a block of $\mathcal{D}$ is maximal in G.*

Proof. Let $\{A,B\}$ be an edge of $J(n,k)$. Then $A \cup B$ has size $k+1$ and so is contained in a unique block Y of $\mathcal{D}$, and hence $\{A,B\}$ is contained in a unique part P_Y of $\mathcal{P}$. Thus $(J(n,k), \mathcal{P})$

is a decomposition. Since G is transitive on $\mathcal{B}$ the pair $(J(n,k),\mathcal{P})$ is G-transitive. Moreover, each P_Y consists of all k-subsets of the l-set Y and so is isomorphic to $J(l,k)$. Since the stabiliser in G of P_Y is G_Y, the last statement follows. □

Construction 2.12. Let $\mathcal{D}=(X,\mathcal{B})$ be an $S(k+1,l,n)$ Steiner system with automorphism group G. Let $i=l-k-1$ and suppose that G is i-transitive on X. For each i-subset Y of X let

$$P_Y=\big\{\{A,B\}\bigm| |A|=|B|=k,\ |A\cap B|=k-1 \text{ and } A\cup B\cup Y\in\mathcal{B}\big\}.$$

Define

$$\mathcal{P}=\{P_Y \mid Y \text{an } i\text{-subset of } X\}.$$

Lemma 2.13. *The pair $(J(n,k),\mathcal{P})$ yielded by Construction* 2.12 *is a G-transitive decomposition with divisors isomorphic to mK_{k+1}, where m is the number of blocks of $\mathcal{D}$ containing an i-set. Moreover, the decomposition is G-primitive if and only if the stabiliser of an i-set is maximal in G.*

Proof. Let $\{A,B\}$ be an edge of $J(n,k)$. Then $A\cup B$ is contained in a unique block W of $\mathcal{D}$ and the unique part of $\mathcal{P}$ containing $\{A,B\}$ is P_Y where $Y=W\backslash(A\cup B)$. Each block containing Y contributes a copy of $J(k+1,k)\cong K_{k+1}$ to P_Y, and since each $(k+1)$-subset is in a unique block, no two blocks containing Y share a vertex of P_Y. Hence the m copies of K_{k+1} in P_Y, are pairwise vertex-disjoint, that is $P_Y\cong mK_{k+1}$. Since G is i-transitive, it follows that $(J(n,k),\mathcal{P})$ is a G-transitive decomposition. Since the stabiliser in G of P_Y is G_Y, the last statement follows. □

Construction 2.14. Let $\mathcal{D}=(X,\mathcal{B})$ be an $S(k+1,k+2,n)$ Steiner system with automorphism group G such that G acts 3-transitively on X. For each 3-subset Y of X, let

$$P_Y=\big\{\big\{Z\cup\{u\},Z\cup\{v\}\big\}\bigm| |Z|=k-1,\ Z\cup Y\in\mathcal{B},\ u,v\in Y\big\}$$

and let $\mathcal{P}=\{P_Y \mid Y \text{ a 3-subset of } X\}$.

Lemma 2.15. *The pair $(J(n,k),\mathcal{P})$ yielded by Construction* 2.14 *is a G-transitive decomposition with divisors isomorphic to mK_3, where m is the number of blocks of $\mathcal{D}$ containing a given* 3*-set. Moreover, the decomposition is G-primitive if and only if the stabiliser of a* 3*-subset is maximal in G.*

Proof. Let $\{A,B\}$ be an edge of $J(n,k)$. Then $A\cup B$ is contained in a unique block W of $\mathcal{D}$ and the unique part of $\mathcal{P}$ containing $\{A,B\}$ is P_Y where $Y=W\backslash(A\cap B)$. Each block containing Y contributes a copy of K_3 to P_Y, and since each $(k+1)$-subset is in a unique block, no two blocks containing Y share a vertex of P_Y. Hence the m copies of K_3 in P_Y are pairwise vertex-disjoint, that is, $P_Y\cong mK_3$. Since G is 3-transitive, it follows that $(J(n,k),\mathcal{P})$ is a G-transitive decomposition. Since the stabiliser in G of P_Y is G_Y, the last statement follows. □

Construction 2.16. Let $\mathcal{D}=(X,\mathcal{B})$ be an $S(k+1,k+2,n)$ Steiner system with k-transitive automorphism group G. For each k-subset Y of X let

$$P_Y=\big\{\big\{\{u\}\cup Z,\{v\}\cup Z\big\}\bigm| Y\cup\{u,v\}\in\mathcal{B},\ Z\subset Y,\ |Z|=k-1\big\}$$

and let $\mathcal{P}=\{P_Y \mid Y \text{ a } k\text{-subset of } X\}$.

Lemma 2.17. *The pair $(J(n,k),\mathcal{P})$ yielded by Construction* 2.16 *is a G-transitive decomposition with divisors isomorphic to mkK_2, where m is the number of blocks of $\mathcal{D}$ containing a given k-set. Moreover, the decomposition is G-primitive if and only if the stabiliser of a k-subset is maximal in G.*

Proof. Let $\{A,B\}$ be an edge of $J(n,k)$. Then $A\cup B$ is contained in a unique block W of $\mathcal{D}$ and the unique part of $\mathcal{P}$ containing $\{A,B\}$ is P_Y where $Y=W\backslash(A\ominus B)$. Each block containing Y contributes a copy of kK_2 to P_Y, and since each $(k+1)$-subset is in a unique block, no two blocks containing Y share a vertex of P_Y. Hence the m copies of kK_2 in P_Y, are pairwise vertex-disjoint, that is $P_Y\cong mkK_2$. Since G is k-transitive, it follows that $(J(n,k),\mathcal{P})$ is a G-transitive decomposition. Since the stabiliser in G of P_Y is G_Y, the last statement follows. □

We end this section with a standard construction of arc-transitive graphs.

Let G be a group with corefree subgroup H and let $g\in G$ such that $g^2\in H$ and $g\notin N_G(H)$. Define the graph $\Gamma=\mathrm{Cos}(G,H,HgH)$ with vertex set the set of right cosets of H in G and Hx adjacent to Hy if and only if $xy^{-1}\in HgH$. Then G acts faithfully and arc-transitively on Γ by right multiplication. We have the following lemma, see for example [16].

Lemma 2.18. *Let Γ be a G-arc-transitive graph with adjacent vertices v and w. Let $H=G_v$, and let $g\in G$ interchange v and w. Then $\Gamma\cong\mathrm{Cos}(G,H,HgH)$. The connected component of Γ containing v consists of all cosets of H contained in $\langle H,g\rangle$. In particular, Γ is connected if and only if $\langle H,g\rangle=G$.*

3. Groups

In this section, we determine the groups G such that $J(n,k)$ is G-vertex-transitive and G-arc-transitive.

Theorem 3.1. *(See [4, Theorem 9.1.2].) Let n,k be positive integers and let $\Gamma=J(n,k)$. If $n>2k$ then $\mathrm{Aut}(\Gamma)=S_n$ with the action induced from the action of S_n on X. For $n=2k\geqslant 4$, $\mathrm{Aut}(\Gamma)=S_n\times S_2=\langle S_n,\tau\rangle$ where τ acts on $V\Gamma$ by complementation in X.*

Given a subset A of X we denote the complement of A in X by $\overline{A}$. Also, if $|X|=n$ and $|A|=k$ then $\Gamma(A)$ denotes the set of neighbours of A in the graph $J(n,k)$, that is, vertices B such that $\{A,B\}$ is an edge.

Lemma 3.2. *(See [11, Proposition 3.2].) Let $\Gamma=J(n,k)$ and $G\leqslant S_n$. The graph Γ is G-arc-transitive if and only if G is k-homogeneous on X and, for a k-subset A, G_A is transitive on $A\times\overline{A}$.*

Proof. Note that G is arc-transitive if and only if G is vertex-transitive and G_A is transitive on $\Gamma(A)$. By definition, Γ is G-vertex-transitive if and only if G is k-homogeneous on X. Moreover, G_A is transitive on $\Gamma(A)$ if and only if G_A is independently transitive on the set of $(k-1)$-subsets of A and on $\overline{A}$, that is, if and only if G_A is transitive on $A\times\overline{A}$. □

Corollary 3.3. *If $G\leqslant S_n$ is $(k+1)$-transitive, then Γ is G-arc-transitive. If Γ is G-arc-transitive and $G\leqslant S_n$, then G is k- and $(k+1)$-homogeneous.*

Theorem 3.4. *Let $n \geqslant 2k \geqslant 4$ and $G \leqslant S_n$. The graph $\Gamma = J(n,k)$ is G-arc-transitive if and only if G is $(k+1)$-transitive on X or $k=3$, $n=9$, and $G = \mathrm{P\Gamma L}(2,8)$.*

Proof. If G is $(k+1)$-transitive, then by Corollary 3.3, Γ is G-arc-transitive. If $k=3$, $n=9$, and $G=\mathrm{P\Gamma L}(2,8)$, then it is easy to check that G is arc-transitive.

Suppose now that Γ is G-arc-transitive. By Corollary 3.3, G is k- and $(k+1)$-homogeneous on X. If G is not $(k+1)$-transitive, then by [26,29] either $2k \leqslant n \leqslant 2k+1$, or $2 \leqslant k \leqslant 3$ and G is one of a small number of groups.

Suppose first that $k=2$. (This is an improvement on the proof of [11, Proposition 3.3].) Since G is 3-homogeneous, it is transitive on X. For $A=\{a,b\}$, Lemma 3.2 implies that G_A is transitive on $A \times \overline{A}$. Therefore using elements of G_A we can map (a,c) onto (a,d) for any $c,d \in \overline{A}$, and so $G_{a,b}$ is transitive on $\overline{A}$. Similarly, $G_{a,c}$ is transitive on $\overline{\{a,c\}}$ for any $c \in \overline{\{a,b\}}$. Hence G_a is transitive on $\overline{\{a\}}$ and so G is 3-transitive on X.

Next suppose that $k=3$. If G is not 4-transitive then either $n=6,7$, or by [26], G is one of $\mathrm{PGL}(2,8)$, $\mathrm{P\Gamma L}(2,8)$ (with $n=9$), or $\mathrm{P\Gamma L}(2,32)$ (with $n=33$). Let $A=\{a,b,c\}$ and suppose that $G \neq \mathrm{P\Gamma L}(2,8)$.

Suppose first that $G=\mathrm{PGL}(2,8)$. Then $G_A \cong S_3$ and $G_{A,a}=C_2$. Hence G does not satisfy the arc-transitivity condition given in Lemma 3.2. Next suppose that $G=\mathrm{P\Gamma L}(2,32)$. Then $|G_{A,a}|=10$ and so again Lemma 3.2 implies that G is not arc-transitive.

If $n=6$, the only 3-homogeneous and 4-homogeneous group which is not 4-transitive is $\mathrm{PGL}(2,5)$. However, this does not satisfy the condition in Lemma 3.2 for arc-transitivity. There are no 3-homogeneous and 4-homogeneous groups of degree 7 which are not 4-transitive.

Next suppose that $k=4$. If G is not 5-transitive, then $n=8$ or 9. Since G is 4-homogeneous and 5-homogeneous, either G is 4-transitive, or G is one of $\mathrm{PGL}(2,8)$, $\mathrm{P\Gamma L}(2,8)$. However, these two groups are not arc-transitive as the stabiliser of a 4-subset A also stabilises a point in $\overline{A}$. The only 4-transitive groups of degree n are A_n and S_n and they are also 5-transitive.

If $k=5$ and G is not 6-transitive, then $n=10$ or 11. Since G is 5-homogeneous it is 5-transitive and so G contains A_n. Thus G is also 6-transitive. Finally, let $k \geqslant 6$. Since G is k-homogeneous it is k-transitive. The only k-transitive groups for $k \geqslant 6$ are A_n and S_n, which are also $(k+1)$-transitive. □

We need a couple of results for the case $n=2k$.

Theorem 3.5. *Let $\Gamma = J(2k,k)$ and suppose that $G \leqslant \mathrm{Aut}(\Gamma) = S_{2k} \times \langle\tau\rangle$ and Γ is G-arc-transitive. Then either $G \cap S_{2k}$ is arc-transitive on Γ, or $k=2$, $G=\langle A_4, (1,2)\tau\rangle$ and $G \cap S_4 = A_4$ has two orbits on arcs.*

Proof. Let $\hat{G} = G \cap S_{2k}$. If $\hat{G}=G$, we are done. Hence we can assume $\hat{G}$ is an index 2 subgroup of G. The graph Γ is connected and is not bipartite, as it contains 3-cycles. It follows that $\hat{G}$ cannot have two orbits on vertices and so $\hat{G}$ is vertex-transitive.

Suppose that $\hat{G}$ is not arc-transitive, and hence has two orbits of equal size on $A\Gamma$. Let $(A,B) \in A\Gamma$. Then $\hat{G}_{(A,B)} \leqslant G_{(A,B)}$ and $|G_A : G_{(A,B)}| = |\Gamma(A)| = k^2 = 2|\hat{G}_A : \hat{G}_{(A,B)}| = |G_A : \hat{G}_{(A,B)}|$. Hence $\hat{G}_{(A,B)} = G_{(A,B)}$ and k is even.

Suppose first that $k \geqslant 6$. Since $\hat{G}$ is transitive on $V\Gamma$, $\hat{G}$ is k-homogeneous and therefore also k-transitive. Hence $A_{2k} \leqslant \hat{G}$, and so $\hat{G}$ is $(k+1)$-transitive. It follows from Theorem 3.4 that $\hat{G}$ is transitive on $A\Gamma$, which is a contradiction. Thus $k=2$ or 4.

If $k = 4$ then $\hat{G}$ is k-homogeneous. The only 4-homogeneous groups of degree 8 contain A_8, and so are also 5-transitive. By Theorem 3.4, $\hat{G}$ is transitive on $A\Gamma$ in this case, and so $k = 2$.

Since $\hat{G}$ is transitive on $V\Gamma$ and $(n, k) = (4, 2)$ we have that 6 divides $|\hat{G}|$. Since $\hat{G}$ is 2-homogeneous it follows that $A_4 \leqslant \hat{G}$. Moreover, S_4 is arc-transitive and so $\hat{G} = A_4$. There are two groups $G \leqslant S_n \times S_2$ such that $\hat{G} = A_4$ and is of index 2 in G, namely $\langle A_4, \tau\rangle$ and $\langle A_4, (1, 2)\tau\rangle$. It is easy to check that the second group is transitive on $A\Gamma$ but not the first one. $\square$

We also have the following theorem about primitivity.

Theorem 3.6. *Let $\Gamma = J(2k, k)$ and $G \leqslant \mathrm{Aut}(\Gamma) = S_{2k} \times \langle\tau\rangle$ such that both G and $G \cap S_{2k}$ are arc-transitive. Suppose that $(\Gamma, \mathcal{P})$ is a G-primitive decomposition. Then $(\Gamma, \mathcal{P})$ is also $(G \cap S_{2k})$-primitive.*

Proof. Let $\hat{G} = G \cap S_{2k}$, let H be the stabiliser in G of a divisor and $\hat{H} = H \cap \hat{G} = H \cap S_{2k}$. We may suppose that $G \neq \hat{G}$. Moreover, as $\hat{G}$ is arc-transitive it acts transitively on $\mathcal{P}$ and so $\hat{G} \nleqslant H$. Since H is maximal in G it follows that $|H : \hat{H}| = 2$.

Suppose first that $G = \hat{G} \times \langle\tau\rangle$. Now $H = \langle\hat{H}, \sigma\tau\rangle$ for some $\sigma \in \hat{G}$. Since $\hat{H} \lhd H$, the element $\sigma\tau$ (and hence also σ) normalises $\hat{H}$ and $\hat{H}$ contains $(\sigma\tau)^2 = \sigma^2$. This implies that $H \leqslant \langle\hat{H}, \sigma\rangle \times \langle\tau\rangle \leqslant G$. Since H is maximal in G, either $H = \langle\hat{H}, \sigma\rangle \times \langle\tau\rangle$ or $\langle\hat{H}, \sigma\rangle \times \langle\tau\rangle = G$. The first implies that $\sigma \in \hat{H}$ and hence $H = \hat{H} \times \langle\tau\rangle$. Thus $\hat{H}$ is maximal in $\hat{G}$ and so by Lemma 2.7, $\mathcal{P}$ is $\hat{G}$-primitive. On the other hand, the second implies $\hat{G} = \langle\hat{H}, \sigma\rangle$. Since $\sigma^2 \in \hat{H}$, we have $|\mathcal{P}| = |\hat{G} : \hat{H}| = 2$ and so again $\hat{G}$ is primitive on $\mathcal{P}$.

Suppose now that $G = \langle\hat{G}, \sigma\tau\rangle$ for some $\sigma \in S_{2k}\backslash\{1\}$ and $\tau \notin G$. Then σ normalises $\hat{G}$ and $\sigma^2 \in \hat{G}$. Also, as $\tau \notin G$, we have $\sigma \notin \hat{G}$ and in particular $\hat{G} \neq S_{2k}$. By Theorem 3.4 and the fact that $n = 2k$, the classification of $(k + 1)$-transitive groups (see for example [6, pp. 194–197]) implies that $\hat{G} = A_{2k}$ and $k \geqslant 3$. Let $\phi : S_{2k} \times \langle\tau\rangle \to S_{2k}$ be the projection of $\mathrm{Aut}(\Gamma)$ onto S_{2k}. Then $\phi_{|G}$ is an isomorphism. Moreover, for an edge $\{A, B\}$ contained in the divisor stabilised by H, $\phi(G_{A,B}) = S_{k-1} \times S_{k-1}$. Since $k \geqslant 3$, there is a transposition in $\phi(G_{A,B})$ and so by [32, Theorem 13.1] and since $\phi(G_{A,B}) \subseteq \phi(H)$, $\phi(H)$ is not primitive. It follows that $\phi(H)$ is a maximal intransitive subgroup of S_{2k} or a maximal imprimitive subgroup of S_{2k} preserving a partition into at most 3 parts. Thus by [28] and since $\hat{H} = \phi(H) \cap A_{2k}$, it follows that $\hat{H}$ is a maximal subgroup of $\hat{G} = A_{2k}$. Hence again $\hat{G}$ is primitive on $\mathcal{P}$. $\square$

4. Alternating and symmetric groups

We have already seen the S_n-transitive decompositions $\mathcal{P}_\cap, \mathcal{P}_\cup$ and $\mathcal{P}_\ominus$. Since $n \geqslant 2k$ it follows that S_n always acts primitively on $\mathcal{P}_\cap$. Also, S_n acts primitively on $\mathcal{P}_\cup$ if and only if $n \neq 2k + 2$. When $n = 2k + 2$, applying Construction 2.1 to $\mathcal{P}_\cup$ we obtain an S_n-primitive decomposition with divisors isomorphic to $2K_{k+1}$. Finally S_n acts primitively on $\mathcal{P}_\ominus$ if and only if $(n, k) \neq (4, 2)$. This justifies the first four lines of Table 2. We also have the following two examples.

Example 4.1.

(1) Let $G = S_4$, $H = \langle(1, 2, 3, 4), (1, 3)\rangle \cong D_8$, $A = \{1, 2\}$ and $B = \{2, 3\}$. Then $P = \{A, B\}^H$ is the 4-cycle

Table 2
S_n-primitive decompositions of $J(n,k)$

$\mathcal{P}$	P	G_P	(n,k)
$\mathcal{P}_\cap$	K_{n-k+1}	$(k-1)$-set stabiliser	
$\mathcal{P}_\cup$	K_{k+1}	$(k+1)$-set stabiliser	$n \neq 2k+2$
$\mathcal{P}_\ominus$	$\binom{n-2}{k-1}K_2$	2-set stabiliser	$(n,k) \neq (4,2)$
$\mathcal{P}_\cup$ and Construction 2.1	$2K_{k+1}$	$S_{k+1} \operatorname{wr} S_2$	$n = 2k+2$
Example 4.1(1)	C_4	D_8	$(n,k) = (4,2)$
Example 4.1(2)	$6K_2$	$S_2 \operatorname{wr} S_3$	$(n,k) = (6,3)$

$$\big\{\big\{\{1,2\},\{2,3\}\big\},\big\{\{2,3\},\{3,4\}\big\},\big\{\{3,4\},\{1,4\}\big\},\big\{\{1,4\},\{1,2\}\big\}\big\}.$$

Since $G_{\{A,B\}} = \langle(1,3)\rangle$ we have $G_{\{A,B\}} < H < G$ and so by Lemma 2.4, $(J(4,2),\mathcal{P})$ is a G-primitive decomposition with $\mathcal{P} = \{P^g \mid g \in G\}$.

(2) Let $G = S_6$ and H be the stabiliser in G of the partition $\{\{1,4\},\{2,3\},\{5,6\}\}$ of $\{1,\ldots,6\}$. Let $A = \{1,2,3\}$ and $B = \{2,3,4\}$. Then $P = \{A,B\}^H$ is the matching

$$\big\{\big\{\{1,2,3\},\{2,3,4\}\big\},\big\{\{2,5,6\},\{3,5,6\}\big\},\big\{\{1,4,5\},\{1,4,6\}\big\},$$
$$\big\{\{1,5,6\},\{4,5,6\}\big\},\big\{\{2,3,5\},\{2,3,6\}\big\},\big\{\{1,4,2\},\{1,4,3\}\big\}\big\}.$$

Since $G_{\{A,B\}} < H < G$ it follows from Lemma 2.4 that $(J(6,3),\mathcal{P})$ is a G-primitive decomposition with $\mathcal{P} = \{P^g \mid g \in G\}$.

We have now constructed all the S_n-primitive decompositions in Table 2. It remains to prove that these are the only ones.

Theorem 4.2. *If $(J(n,k),\mathcal{P})$ is an S_n-primitive decomposition with $n \geqslant 2k$ then $\mathcal{P}$ is given by one of the rows of Table* 2.

Proof. Let $\Gamma = J(n,k)$, $X = \{1,\ldots,n\}$, and let $A = \{1,2,\ldots,k\}$ and $B = \{2,\ldots,k+1\}$ be adjacent vertices of Γ. Then $G_{\{A,B\}} = \mathrm{Sym}(\{1,k+1\}) \times \mathrm{Sym}(\{2,\ldots,k\}) \times \mathrm{Sym}(\{k+2,\ldots,n\})$. By Lemma 2.4, to find all G-primitive decompositions of Γ, we need to determine all maximal subgroups H of G which contain $G_{\{A,B\}}$. Since $G_{\{A,B\}}$ contains a 2-cycle, [32, Theorem 13.1] implies that there are no proper primitive subgroups of G containing $G_{\{A,B\}}$. Hence H is either imprimitive or intransitive.

Suppose first that H is intransitive. Then H is a maximal intransitive subgroup and hence it has two orbits U, W on X and $H = \mathrm{Sym}(U) \times \mathrm{Sym}(W)$. Since $G_{\{A,B\}} \leqslant H$, the only possibilities for these two orbits are:

$$\begin{array}{lll}
\{1,\ldots,k+1\}, & \{k+2,\ldots,n\}, & n \neq 2k+2, \\
\{1,k+1\}, & X\backslash\{1,k+1\}, & (n,k) \neq (4,2), \\
\{2,\ldots,k\}, & \{1,k+1,k+2,\ldots,n\}. &
\end{array}$$

When $H = \mathrm{Sym}(\{1,\ldots,k+1\}) \times \mathrm{Sym}(\{k+2,\ldots,n\}) = G_{A\cup B}$, we obtain the decomposition $(\Gamma,\mathcal{P}_\cup)$, while $H = \mathrm{Sym}(\{1,k+1\}) \times \mathrm{Sym}(X\backslash\{1,k+1\}) = G_{A\ominus B}$ yields the decomposition $(\Gamma,\mathcal{P}_\ominus)$. Finally, $H = \mathrm{Sym}(\{2,\ldots,k\}) \times \mathrm{Sym}(\{1,k+1,k+2,\ldots,n\}) = G_{A\cap B}$ gives us the decomposition $(\Gamma,\mathcal{P}_\cap)$.

If H is transitive but imprimitive, then the possible systems of imprimitivity are:

$\{1,\dots,k+1\},\{k+2,\dots,2k+2\}$	when $n=2k+2$,
$\{1,4\},\{2,3\},\{5,6\}$	when $(n,k)=(6,3)$,
$\{1,3\},\{2,4\}$	when $(n,k)=(4,2)$.

In the first case, $P=\{A,B\}^H$ is the union of two cliques each of size $k+1$, and has as vertices all k-subsets of $\{1,\dots,k+1\}$ and all k-subsets of $\{k+2,\dots,2k+2\}$, that is we get the decomposition obtained from applying Construction 2.1 to $\mathcal{P}_\cup$. The last two cases give us the two decompositions from Example 4.1. $\square$

By Theorem 3.4, A_n is arc-transitive on $J(n,k)$ if and only if $n \geqslant 5$. Moreover, all the S_n-primitive decompositions in Table 2 are A_n-primitive decompositions. We have the following extra examples for alternating groups.

Example 4.3.

(1) Let $(n,k)=(5,2)$, $G=A_5$, $A=\{1,2\}$ and $B=\{2,3\}$. Then $G_{\{A,B\}}=\langle(1,3)(4,5)\rangle$ and is contained in the maximal subgroup $H=\langle(1,2,3,4,5),(1,3)(4,5)\rangle\cong D_{10}$ of G. Letting $P=\{A,B\}^H$ and $\mathcal{P}=\{P^g \mid g\in G\}$, Lemma 2.4 implies that $(J(5,2),\mathcal{P})$ is an A_5-primitive decomposition. Since $H_A\cong C_2$ it follows that the divisors are cycles of length 5.

(2) Let $(n,k)=(6,3)$, $G=A_6$, $A=\{1,2,3\}$ and $B=\{2,3,4\}$. Then $G_{\{A,B\}}=\langle(2,3)(5,6),(1,4)(5,6)\rangle$ and is contained in the maximal subgroup $H=\langle(2,3)(5,6),(1,4,5)(2,3,6)\rangle\cong \mathrm{PSL}(2,5)$ of G. Letting $P=\{A,B\}^H$ and $\mathcal{P}=\{P^g \mid g\in G\}$, Lemma 2.4 implies that $(J(6,3),\mathcal{P})$ is an A_6-primitive decomposition. Now P is a graph on 10 vertices with valency 3 admitting an arc-transitive action of $H\cong A_5$. Hence P is the Petersen graph.

Lemma 4.4. *Let $\mathcal{P}$ be the decomposition of $J(6,3)$ given by Example* 4.3(2). *Then $\mathcal{P}$ is G-primitive if and only if $G=A_6$ or $\langle A_6,(1,2)\tau\rangle$ where τ is the complementation map as in Theorem* 3.1.

Proof. As in the example, we take $A=\{1,2,3\}$, $B=\{2,3,4\}$ and $P=\{A,B\}^H$ for $H=\langle(2,3)(5,6),(1,4,5)(2,3,6)\rangle\cong A_5$.

If $G\leqslant S_6$, by Theorem 3.4, G must be 4-transitive, so $A_6\leqslant G$. We have seen above that $\mathcal{P}$ is A_6-primitive. However, S_6 does not preserve the partition $\mathcal{P}$ of Example 4.3(2), since $(1,4)$ preserves $\{A,B\}$ but not P. So assume $G\nleqslant S_6$. By Theorems 3.5 and 3.6, $\mathcal{P}$ is a $(G\cap S_6)$-primitive decomposition. Thus $G\cap S_6=A_6$ and so $G=G_1=\langle A_6,\tau\rangle$ or $G=G_2=\langle A_6,(1,2)\tau\rangle$. Thus $|G|=2|A_6|$ and so $|G_P:H|=2$. Then as $G_{\{A,B\}}\leqslant G_P$ it follows that $G_{\{A,B\}}$ normalises H. However, $(2,5)(3,6)\tau\in(G_1)_{\{A,B\}}$ and does not normalise H, so $G\neq G_1$. Now $(G_2)_{\{A,B\}}=\langle(1,4)(2,5)(3,6)\tau, H_{\{A,B\}}\rangle$ does normalise H and so fixes P. Thus $\langle H,(1,4)(2,5)(3,6)\tau\rangle=(G_2)_P\cong S_5$ which is a maximal subgroup of $G_2\cong S_6$. Hence $\mathcal{P}$ is a G_2-primitive decomposition. $\square$

We now show that Example 4.3 yields the only A_n-primitive decompositions which are not S_n-primitive.

Theorem 4.5. *Let $(J(n,k),\mathcal{P})$ be an A_n-primitive decomposition such that A_n is arc-transitive and $n \geqslant 2k$. Then $\mathcal{P}$ is either an S_n-primitive decomposition, or $(n,k)=(5,2)$ or $(6,3)$ and $\mathcal{P}$ is isomorphic to a decomposition given in Example* 4.3.

Proof. Let $\Gamma = J(n,k)$. Since $G = A_n$ is arc-transitive it follows from Theorem 3.4 that $n \geqslant 5$. Let $X = \{1,\dots,n\}$, $A = \{1,\dots,k\}$ and $B = \{2,\dots,k+1\}$. Then

$$G_{\{A,B\}} = \big(\mathrm{Sym}\big(\{1,k+1\}\big) \times \mathrm{Sym}\big(\{2,\dots,k\}\big) \times \mathrm{Sym}\big(\{k+2,\dots,n\}\big)\big) \cap A_n.$$

We need to consider all maximal subgroups H such that $G_{\{A,B\}} < H < G$. For each such H, $P = \{A,B\}^H$ is the edge-set of a divisor of the G-primitive decomposition.

Suppose first that H is intransitive on X. Then $G_{\{A,B\}}$ has the same orbits on X as $(S_n)_{\{A,B\}}$ and so H is the intersection with A_n of one of the maximal intransitive subgroups which we considered in the S_n case in the proof of Theorem 4.2. Moreover, we obtain the decompositions in rows 1–3 in Table 2, and so $(\Gamma,\mathcal{P})$ is S_n-primitive.

Next suppose that H is imprimitive on X. Since $G_{\{A,B\}}$ is primitive on both $A \cap B$ and $\overline{A \cup B}$, the only systems of imprimitivity preserved by $G_{\{A,B\}}$ are those discussed in the S_n case. Thus H is the intersection with A_n of one of the maximal imprimitive subgroups considered in the S_n case and we obtain the decompositions in rows 4 and 6 in Table 2. Thus $(\Gamma,\mathcal{P})$ is S_n-primitive.

Finally, suppose that H is primitive on X. If $k-1 \geqslant 3$ or $n-k-1 \geqslant 3$, the edge stabiliser $G_{\{A,B\}}$, and hence H, contains a 3-cycle. Hence by [32, Theorem 13.3], $H = A_n$, contradicting H being a proper subgroup. Note that if $k \geqslant 4$ then $k-1 \geqslant 3$, and so (n,k) is one of $(5,2)$ or $(6,3)$.

If $(n,k)=(5,2)$ then $G_{\{A,B\}} = \langle (1,3)(4,5)\rangle$ and $H \cong D_{10}$. Since A_5 contains 15 involutions, D_{10} contains 5 involutions and there are six subgroups D_{10} in A_5, it follows that there are 2 choices for H and these are

$$H_1 = \big\langle (1,2,3,4,5),(1,3)(4,5)\big\rangle,$$
$$H_2 = \big\langle (1,4,5,3,2),(1,3)(4,5)\big\rangle.$$

Note that $H_2 = H_1^{(1,3)}$ and $(1,3) \in (S_n)_{\{A,B\}}$ and so by Lemma 2.6 the two decompositions obtained are isomorphic. Moreover, H_1 is the stabiliser of the divisor containing $\{A,B\}$ in the decomposition in Example 4.3(1).

If $(n,k)=(6,3)$ then $G_{\{A,B\}} = \langle (2,3)(5,6),(1,4)(5,6)\rangle$ and $H \cong \mathrm{PSL}(2,5)$. A computation using MAGMA [3] showed that there are two choices for H containing $G_{\{A,B\}}$ and these are:

$$H_1 = \big\langle (2,3)(5,6),(1,4,5)(2,3,6)\big\rangle,$$
$$H_2 = \big\langle (2,3)(5,6),(1,4,5)(3,2,6)\big\rangle.$$

Note that $H_2 = H_1^{(2,3)}$ and $(2,3) \in (S_n)_{\{A,B\}}$ and so the two decompositions obtained are isomorphic. Moreover, H_1 is the stabiliser of the divisor containing $\{A,B\}$ in the decomposition in Example 4.3(2). □

We now look at the case where $n = 2k$ and G is not a subgroup of S_n.

Example 4.6. Let $(n,k)=(4,2)$ and $G = \langle A_4,(1,2)\tau\rangle$. Let $A = \{1,2\}$ and $B = \{2,3\}$. Then $G_{\{A,B\}} = \langle (2,4)\tau\rangle$.

(1) Let $H_1 = \langle (1,2,4), (1,2)\tau\rangle$ and

$$P = \{A, B\}^{H_1} = \bigl\{\bigl\{\{1,2\},\{2,3\}\bigr\}, \bigl\{\{2,4\},\{3,4\}\bigr\}, \bigl\{\{1,4\},\{1,3\}\bigr\}\bigr\}.$$

Since $G_{\{A,B\}} \leqslant H_1$, it follows from Lemma 2.4 that $(J(4,2), P^G)$ is a G-primitive decomposition, with divisors isomorphic to $3K_2$.

(2) Let $H_2 = \langle (1,2)(3,4), (1,3)(2,4), (1,3)\tau\rangle$ and

$$P = \{A, B\}^{H_2} = \bigl\{\bigl\{\{1,2\},\{2,3\}\bigr\}, \bigl\{\{2,3\},\{3,4\}\bigr\}\bigl\{\{3,4\},\{1,4\}\bigr\}, \bigl\{\{1,4\},\{1,2\}\bigr\}\bigr\}.$$

Since $G_{\{A,B\}} \leqslant H_2$, it follows from Lemma 2.4 that $(J(4,2), P^G)$ is a G-primitive decomposition, with divisors isomorphic to C_4. Notice that this decomposition is the one in Example 4.1(1) and so is also S_4-primitive.

Theorem 4.7. *Let $\Gamma = J(n,k)$ with $n = 2k$ and let $G \leqslant \mathrm{Aut}(\Gamma) = S_n \times S_2$ such that G is not contained in S_n. Further, suppose that $(\Gamma, \mathcal{P})$ is a G-primitive decomposition which is not $(G \cap S_n)$-primitive. Then $n = 4$ and $\mathcal{P}$ is isomorphic to a decomposition given by Example* 4.6.

Proof. By Theorems 3.5 and 3.6, it follows that $k = 2$ and $G = \langle A_4, (1,2)\tau\rangle$, where τ is complementation in X. Let $A = \{1,2\}$ and $B = \{2,3\}$. Then $G_{\{A,B\}} = \langle (2,4)\tau\rangle$. It is not hard to see that the only maximal subgroups of G containing $G_{\{A,B\}}$ are the groups H_1 and H_2 from Example 4.6, and $H_3 = \langle (2,3,4), (2,3)\tau\rangle$. The first two give the two decompositions from Example 4.6. Note that $(1,3)$ stabilises $\{A,B\}$ and normalises G, and $H_3 = H_1^{(1,3)}$. So by Lemma 2.6, this yields a decomposition isomorphic to the one in Example 4.6(1). □

5. The case $k \geqslant 4$

By Theorem 3.4, if $k \geqslant 4$ then $G \leqslant S_n$ is arc-transitive on $J(n,k)$ if and only if G is $(k+1)$-transitive on the n-set X. Hence by the classification of finite 2-transitive permutation groups, other than A_n or S_n, the only possibilities for (n, G) when $k \geqslant 4$ are $(12, M_{12})$ and $(24, M_{24})$ with $k = 4$.

First we state the following well known lemmas.

Lemma 5.1. *Let $(X, \mathcal{B})$ be the Witt design $S(5,6,12)$. Then $\mathcal{B}$ contains* 132 *elements, called* hexads. *Each point of X is contained in* 66 *hexads, each* 2*-subset in* 30 *hexads, each* 3*-subset in* 12 *hexads, each* 4*-subset in* 4 *hexads, and each* 5*-subset in a unique hexad.*

Proof. The number of hexads is given in [10, p. 31] and then the number of hexads containing a given i-subset is calculated by counting i-subset–hexad pairs in two different ways. □

Lemma 5.2. *(See [24, Lemma 2.11.7].) Suppose that $(X, \mathcal{B})$ is a Witt design $S(5,6,12)$ preserved by $G = M_{12}$ and let $h \in \mathcal{B}$ be a hexad. Then $G_h \cong S_6$ and the actions of G_h on h and $X \backslash h$ are the two inequivalent actions of S_6 on six points.*

Since the stabiliser of a 3-set or a 2-set is maximal in $G = M_{12}$, it follows from Lemma 2.9 that $\mathcal{P}_\cap$ and $\mathcal{P}_\ominus$ are G-primitive decompositions. Moreover, as G acts primitively on the point set X of the Witt design, Construction 2.12 yields a G-primitive decomposition of $J(12,4)$. We also obtain a G-primitive decomposition from Construction 2.14 as G acts primitively on 3-subsets and one from Construction 2.16 as G acts primitively on 4-subsets. The G-transitive decomposition obtained from Construction 2.10 is not primitive as the stabiliser of a hexad is contained in the stabiliser of a pair of complementary hexads. However, applying Construction 2.1 we obtain a G-primitive decomposition with divisors isomorphic to $2J(6,4)$.

Before giving several more constructions arising from the Witt design, we need the following definition and lemma.

Definition 5.3. A *linked three* in $S(5,6,12)$ is a set of four triads (or 3-sets) such that the union of any two is a hexad.

Lemma 5.4. *Let A, B be two triads whose union is a hexad. Then there exists a unique linked three containing both A and B.*

Proof. By Lemma 5.1, there are exactly 12 hexads containing A. If such a hexad contains at least two points of B, then it is $A \cup B$. Let $b \in B$. Then there are 4 hexads containing A and b, and so exactly 3 hexads meet $A \cup B$ in $A \cup \{b\}$. Therefore there are 9 hexads containing A and meeting $A \cup B$ in a 4-set. Hence only two hexads contain A and are disjoint from B. These yield two triads, C and D, forming hexads with A. By Lemma 5.2, the stabiliser of A and B is $S_3 \times S_3$ which acts transitively on the remaining 6 points. Hence C and D must be disjoint. Since the complement of a hexad is a hexad, C and D must form hexads with B too. It follows that $\{A, B, C, D\}$ is the unique linked three containing A and B. $\square$

Construction 5.5. Let $(X, \mathcal{B})$ be the Witt design $S(5,6,12)$ and let $G = M_{12}$.

(1) Let T be a linked three as in Definition 5.3. Let

$$P_T = \big\{ \big\{ \{u\} \cup Y, \{v\} \cup Y \big\} \bigm| Y \in T,\ \{u,v\} \text{ contained in some triad of } T \setminus Y \big\}$$

and $\mathcal{P} = \{P_T \mid T \text{ is a linked three}\}$. Then $P_T \cong 12K_3$, with each triad contributing $3K_3$. If $\{A, B\}$ is an edge of $J(12,4)$ then $A \cup B$ is contained in a unique hexad $A \cup B \cup \{x\}$ for some $x \in X$, and by Lemma 5.4, $\{A \cap B, \{x\} \cup (A \ominus B)\}$ is contained in a unique linked three T. For this T, P_T is the unique part of $\mathcal{P}$ containing $\{A, B\}$. Since G acts transitively on the set of linked threes and the stabiliser of a linked three is maximal, $(J(12,4), \mathcal{P})$ is a G-primitive decomposition.

(2) Let T be a linked three. A 4-set intersecting each triad of T in a single point and such that its union with any triad is a hexad is called *special* for T. For fixed triads T_1, T_2 of T and points $x_1 \in T_1$, $x_2 \in T_2$, these conditions imply that there is at most one special 4-set containing $\{x_1, x_2\}$ and existence of such a 4-set was confirmed by MAGMA [3]. Thus there are nine special 4-sets for T. Let

$$P_T = \big\{ \big\{ \{u,x,y,z\}, \{v,x,y,z\} \big\} \bigm| \{x,y,z,t\} \text{ special 4-set for } T,\ \{u,v,t\} \in T \big\}$$

and $\mathcal{P} = \{P_T \mid T \text{ is a linked three}\}$. Then $P_T \cong 36K_2$, with each special 4-set contributing $4K_2$. If $\{A, B\}$ is an edge of $J(12,4)$ then $A \cup B$ is contained in a unique hexad $A \cup B \cup \{x\}$ for some $x \in X$, and there is a unique linked three T such that $(A \cap B) \cup \{x\}$ is special for T and $\{x\} \cup (A \ominus B)$ is a triad of T (a MAGMA [3] calculation). Thus P_T is the only part of $\mathcal{P}$ containing $\{A, B\}$. Since G acts transitively on the set of linked threes and the stabiliser of a linked three is maximal, $(J(12,4), \mathcal{P})$ is a G-primitive decomposition.

Table 3
M_{12}-primitive decompositions of $J(12,4)$

$\mathcal{P}$	P	G_P
$\mathcal{P}_\cap$	K_9	$M_9 \rtimes S_3$
$\mathcal{P}_\ominus$	$\binom{10}{3}K_2$	$M_{10}.2$
Constructions 2.10 and 2.1	$2J(6,4)$	$M_{10}.2$
Construction 2.12	$66K_5$	M_{11}
Construction 2.14	$12K_3$	$M_9 \rtimes S_3$
Construction 2.16	$16K_2$	$M_8 \rtimes S_4$
Construction 5.5(1)	$12K_3$	$M_9 \rtimes S_3$
Construction 5.5(2)	$36K_2$	$M_9 \rtimes S_3$
Construction 5.6	Σ	M_{11}

Construction 5.6. Let $G = M_{12} < S_{12}$ and let $H = M_{11}$ be a 3-transitive subgroup of G. Then H has an orbit of length 165 on 4-subsets and this orbit forms a $3-(12,4,3)$ design. Let Σ be the subgraph of $J(12,4)$ induced on the orbit of length 165. The graph Σ was studied in [13], where it is seen that Σ has valency 8, is H-arc-transitive and given an edge $\{A,B\}$ we have $H_{\{A,B\}} \cong S_2 \times S_3 = G_{\{A,B\}}$. Thus Lemma 2.4 and the fact that H is maximal in G, imply that $\mathcal{P} = \Sigma^G$ is a G-primitive decomposition of $J(12,4)$.

We have now seen all the M_{12}-primitive decompositions listed in Table 3. It remains to prove that these are the only ones.

Proposition 5.7. *If $(J(12,4),\mathcal{P})$ is an M_{12}-primitive decomposition then $\mathcal{P}$ is given by one of the rows of Table* 3.

Proof. Let $\Gamma = J(12,4)$ and $G = M_{12}$ acting on the point set X of the Witt-design $S(5,6,12)$. Take adjacent vertices $A = \{1,2,3,4\}$ and $B = \{2,3,4,5\}$ and suppose that $h = \{1,2,3,4,5,6\}$ is the unique hexad containing $A \cup B$. Then $G_{\{A,B\}} = G_{\{1,5\},\{2,3,4\},\{6\}} \cong S_2 \times S_3$, by Lemma 5.2. Since transpositions in the action of G_h on h act as a product of three transpositions on $X \backslash h$, and 3-cycles on h act as a product of two 3-cycles on $X \backslash h$ it follows that $G_{1,5,6,\{2,3,4\}} \cong S_3$ acts regularly on $X \backslash h$, and so $G_{\{A,B\}}$ acts transitively on $X \backslash h$.

Let H be a maximal subgroup of G such that $G_{\{A,B\}} \leqslant H < G$. The maximal subgroups of G are given in [10, p. 33]. The orbit lengths of $G_{\{A,B\}}$ imply that $G_{\{A,B\}}$ does not preserve a system of imprimitivity on X with blocks of size 2 or 4 and so $H \ncong C_4^2 \rtimes D_{12}$, $A_4 \times S_3$, or $C_2 \times S_5$. Moreover, $|H_6|$ is even and so $H \ncong \mathrm{PSL}(2,11)$.

If H is intransitive then H is one of $G_{\{2,3,4,6\}}$, $G_{\{2,3,4\}}$, $G_{\{1,5,6\}}$, $G_{\{1,5\}}$ or G_6. (Note that G_h is not maximal.) The first is the stabiliser of the divisor containing $\{A,B\}$ in the decomposition yielded by Construction 2.16. The second gives $\mathcal{P}_\cap$ while the third is the stabiliser of the divisor of the decomposition yielded by Construction 2.14 containing $\{A,B\}$. If $H = G_{\{1,5\}}$ then we obtain the decomposition $\mathcal{P}_\ominus$ while if $H = G_6$ we obtain the decomposition yielded by Construction 2.12.

The only hexad pair fixed by $G_{\{A,B\}}$ is $\{h, X \backslash h\}$. Now G_h is the stabiliser of the divisor of the decomposition yielded by Construction 2.10 containing $G_{\{A,B\}}$. Such a divisor is isomorphic to $J(6,4)$ and so $G_{\{h,X \backslash h\}}$ yields the decomposition with divisors isomorphic to $2J(6,4)$ obtained after applying Construction 2.1.

Table 4
M_{24}-primitive decompositions of $J(24,4)$

$\mathcal{P}$	P	G_P
$\mathcal{P}_\cap$	K_{21}	$\mathrm{P\Gamma L}(3,4)$
$\mathcal{P}_\ominus$	$\binom{22}{3}K_2$	$M_{22}.2$
Construction 2.10	$J(8,4)$	$C_2^4 \rtimes A_8$
Construction 2.12	$21K_5$	$\mathrm{P\Gamma L}(3,4)$

A calculation using MAGMA [3] shows that there is only one transitive subgroup of G isomorphic to M_{11} which contains $G_{\{A,B\}}$ and this yields Construction 5.6.

By the list of maximal subgroups of G given in [10, p. 33], the only case left to consider is H being the stabiliser of a linked three. If T is a linked three preserved by $G_{\{A,B\}}$ then $\{1,5,6\}$ is a triad of T and either $\{2,3,4\}$ is also a triad or 2, 3, and 4 lie in distinct triads. Since a linked three is uniquely determined by any two of its triads (Lemma 5.4), there is a unique linked three T containing $\{1,5,6\}$ and $\{2,3,4\}$. Then G_T is the stabiliser of the divisor of the decomposition yielded by Construction 5.5(1) containing $\{A,B\}$. If 2, 3 and 4 are in distinct blocks, a calculation using MAGMA [3] shows that there is a unique H containing $G_{\{A,B\}}$ and we obtain the decomposition in Construction 5.5(2). □

We need the following well known lemma to deal with the case where $G = M_{24}$.

Lemma 5.8. *(See [24, Lemma 2.10.1].) Let $(X,\mathcal{B})$ be the Witt design $S(5,8,24)$. Then $\mathcal{B}$ contains 759 elements, called* octads. *Each point of X is contained in 253 octads, each 2-subset in 77 octads, each 3-subset in 21 octads, each 4-subset in 5 octads, and each 5-subset in a unique octad. Moreover, the stabiliser of an octad in M_{24} is $C_2^4 \rtimes A_8$ where C_2^4 acts trivially on the octad and transitively on its complement.*

Proof. The number of octads comes from [24, Lemma 2.10.1] and then the numbers of octads containing a given i-subset follows from basic counting. The statement about the stabiliser of an octad also comes from [24, Lemma 2.10.1]. □

Since the stabilisers of a 3-set, of a 2-set, and of an octad are maximal in G, applying Constructions 2.8, 2.10 and 2.12, we get the list of M_{24}-primitive decompositions in Table 4.

Proposition 5.9. *If $(J(24,4),\mathcal{P})$ is an M_{24}-primitive decomposition then $\mathcal{P}$ is given by one of the rows in Table 4.*

Proof. Let $\Gamma = J(24,4)$ and $G = M_{24}$ acting on the point set X of the Witt-design $S(5,8,24)$. Take adjacent vertices $A = \{1,2,3,4\}$ and $B = \{2,3,4,5\}$ and suppose that $\Delta = \{1,2,3,4,5,6,7,8\}$ is the unique octad containing $A \cup B$. Then looking at the stabiliser of an octad given in Lemma 5.8, we see that $G_{\{A,B\}} = G_{\{1,5\},\{2,3,4\},\{6,7,8\}} = C_2^4 \rtimes ((S_2 \times S_3^2) \cap A_8)$ with orbits in Δ of lengths 2, 3, 3. Since $G_{\{A,B\}}$ contains the pointwise stabiliser of the octad Δ, which by Lemma 5.8 acts regularly on $X \setminus \Delta$, it follows that $G_{\{A,B\}}$ is transitive on $X \setminus \Delta$.

Let H be a maximal subgroup of G such that $G_{\{A,B\}} \leqslant H < G$. The maximal subgroups of G are given in [10, p. 96], and comparing orders we see that $H \not\cong \mathrm{PSL}(2,7)$ or $\mathrm{PSL}(2,23)$. Since $G_{\{A,B\}}$ has an orbit of length 16 and an orbit of length 3 in X, it cannot fix a pair of dodecads. Similarly, if H fixed a trio of disjoint octads, one of the three octads would be Δ and $G_{\{A,B\}}$ would interchange the other 2. However, all index 2 subgroups of $G_{\{A,B\}}$ are transitive on $X \setminus \Delta$ (a MAGMA calculation [3]) and so H does not fix a trio of disjoint octads. Suppose next that H fixes a sextet, that is, 6 sets of size 4 such that the union of any two is an octad. Then the $G_{\{A,B\}}$-orbit $X \setminus \Delta$ is the union of four of these sets. However, the remaining $G_{\{A,B\}}$-orbit lengths are incompatible with H fixing a partition of $\{1,\dots,8\}$ into two sets of size 4. Thus the list of maximal subgroups of G in [10, p. 96] implies that H is intransitive on X, and so $H = G_{\{1,5\}}, G_{\{2,3,4\}}, G_{\{6,7,8\}}$, or $G_{\{1,2,3,4,5,6,7,8\}}$. By Lemma 2.9, the first gives the decomposition $\mathcal{P}_{\ominus}$ while the second gives $\mathcal{P}_{\cap}$. The third is the stabiliser of the divisor of the decomposition yielded by Construction 2.12 containing $\{A,B\}$ while the fourth yields the decomposition obtained from Construction 2.10. $\square$

6. The case $k = 3$

By Theorem 3.4, $G \leqslant S_n$ is arc-transitive on $J(n,3)$ if and only if G is 4-transitive or $G = \mathrm{P\Gamma L}(2,8)$ and $n = 9$. Thus other than A_n or S_n the only possibilities for (n,G) are $(11, M_{11})$, $(12, M_{12})$, $(23, M_{23})$, $(24, M_{24})$ and $(9, \mathrm{P\Gamma L}(2,8))$.

Since the stabiliser of a 2-subset is maximal in M_{24}, it follows that $\mathcal{P}_{\cap}$ and $\mathcal{P}_{\ominus}$ are M_{24}-primitive decompositions with divisors K_{22} and $\binom{22}{2}K_2$ respectively. We also have a construction involving sextets.

Construction 6.1. Let S be a sextet, that is, a set of six 4-subsets such that the union of any two is an octad, and define $P_S = \{\{A,B\} \mid A \cup B \in S\}$ and $\mathcal{P} = \{P_S \mid S \text{ a sextet}\}$. Then $P_S \cong 6J(4,3) \cong 6K_4$ with one copy of K_4 for each 4-set in S. Let $\{A,B\}$ be an edge of $J(24,3)$. By [24, Lemma 2.3.3], $A \cup B$ is a member of a unique sextet S and so P_S is the only part of $\mathcal{P}$ containing $\{A,B\}$. Since G acts primitively on the set of sextets, it follows that $(J(24,3),\mathcal{P})$ is an M_{24}-primitive decomposition.

Proposition 6.2. *If $(J(24,3),\mathcal{P})$ is an M_{24}-primitive decomposition then either $\mathcal{P} = \mathcal{P}_{\ominus}$ or $\mathcal{P}_{\cap}$, or $\mathcal{P}$ arises from Construction* 6.1.

Proof. Let $\Gamma = J(24,3)$ and $G = M_{24}$ acting on the point set X of the Witt-design $S(5,8,24)$. Let $A = \{1,2,3\}$ and $B = \{2,3,4\}$ be adjacent vertices in Γ. Then $G_{\{A,B\}} = G_{\{1,4\},\{2,3\}}$ which is the stabiliser in $\mathrm{Aut}(M_{22})$ of a 2-subset and so by [10, p. 39], $G_{\{A,B\}} \cong 2^5 \rtimes S_5$. Since G is 5-transitive on X, $G_{\{A,B\}}$ is transitive on $X \setminus \{1,2,3,4\}$.

Let H be a maximal subgroup of G such that $G_{\{A,B\}} \leqslant H < G$. The maximal subgroups of G can be found in [10]. Comparing orders we see that $H \ncong \mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,23)$, or the stabiliser of a trio of distinct octads. Now $G_{\{A,B\}}$ contains $G_{1,2,3,4}$ which is transitive on the remaining 20 points. Thus $G_{1,2,3,4}$ does not fix a pair of dodecads and so neither does H. Hence by the list of maximal subgroups of G in [10, p. 96], either H is intransitive, or fixes a sextet. If H is intransitive, then $H = G_{\{1,4\}}$ or $G_{\{2,3\}}$. By Lemma 2.9, the first gives $\mathcal{P}_{\ominus}$ while the second gives $\mathcal{P}_{\cap}$.

Suppose then that H fixes a sextet. The orbit lengths of $G_{\{A,B\}}$ imply that $\{1,2,3,4\}$ is one of the blocks of the sextet. By [24, Lemma 2.3.3], $\{1,2,3,4\}$ is contained in a unique sextet S. Thus $H = G_S$ and is the stabiliser in G of the divisor of the decomposition obtained from Construction 6.1 containing $\{A,B\}$. $\square$

Table 5
M_{23}-primitive decompositions of $J(23,3)$

$\mathcal{P}$	P	G_P
$\mathcal{P}_\cap$	K_{21}	$\mathrm{P\Sigma L}(3,4)$
$\mathcal{P}_\ominus$	$\binom{21}{2}K_2$	$\mathrm{P\Sigma L}(3,4)$
Construction 2.10	$J(7,3)$	$C_2^4 \rtimes A_7$
Construction 2.12	$5K_4$	$C_2^4 \rtimes (C_3 \times A_5) \rtimes C_2$

Before dealing with $G = M_{23}$ we need the following well known result which follows from Lemma 5.8.

Lemma 6.3. *Let $(X, \mathcal{B})$ be the Witt design $S(4,7,23)$. Then $\mathcal{B}$ contains* 253 *elements, called* heptads. *Each point of X is contained in* 77 *heptads, each* 2*-subset in* 21 *heptads, each* 3*-subset in* 5 *heptads, and each* 4*-subset in a unique heptad. Moreover, the stabiliser of a heptad is $C_2^4 \rtimes A_7$ with the pointwise stabiliser of the heptad being C_2^4 which acts regularly on the* 16 *points not in the heptad.*

Proof. Since $(X, \mathcal{B})$ is derived from the set of all blocks of the Witt design $S(5,8,24)$ containing a given point, this follows from Lemma 5.8. $\square$

Using the Witt design $S(4,7,23)$ and the fact that the stabiliser of a 2-set is maximal in M_{23} we get the M_{23}-primitive decompositions in Table 5. These are in fact all such decompositions.

Proposition 6.4. *If $(J(23,3), \mathcal{P})$ is an M_{23}-primitive decomposition then $\mathcal{P}$ is as in one of the lines of Table* 5.

Proof. Let $\Gamma = J(23,3)$ and $G = M_{23}$ acting on the point set X of the Witt-design $S(4,7,23)$. Take adjacent vertices $A = \{1,2,3\}$ and $B = \{2,3,4\}$. By Lemma 6.3, $\{1,2,3,4\}$ is contained in a unique heptad, $h = \{1,2,3,4,5,6,7\}$ say, and so $G_{\{A,B\}}$ fixes h. Since the stabiliser of a heptad is isomorphic to $C_2^4 \rtimes A_7$ (Lemma 6.3), it follows that $G_{\{A,B\}}$ has order 192 and has orbits $\{1,4\}$, $\{2,3\}$, $\{5,6,7\}$ and $X \backslash h$.

Let H be a maximal subgroup of G such that $G_{\{A,B\}} \leqslant H < G$. The maximal subgroups of G can be found in [10]. By comparing orders, $H \ncong C_{23} \rtimes C_{11}$ and so H is intransitive. Thus $H = G_{\{1,4\}}, G_{\{2,3\}}, G_{\{5,6,7\}}$ or G_h. By Lemma 2.9, the first two give the decompositions $\mathcal{P}_\ominus$ and $\mathcal{P}_\cap$ respectively. Also $G_{\{5,6,7\}}$ is the stabiliser of the divisor of the decomposition obtained from Construction 2.12 containing $\{A, B\}$ while G_h is the stabiliser of the divisor of the decomposition yielded by Construction 2.10. $\square$

Since 4-set stabilisers and 2-set stabilisers are maximal in M_{12}, it follows from Lemma 2.9 that $\mathcal{P}_\cup$, $\mathcal{P}_\cap$ and $\mathcal{P}_\ominus$ are M_{12}-primitive decompositions with divisors isomorphic to K_4, K_{10} and $\binom{10}{2}K_2$ respectively. We also have the following construction.

Construction 6.5. Let $(X, \mathcal{B})$ be the Witt design $S(5,6,12)$. Let F be a *linked four*, that is a set of three mutually disjoint tetrads (sets of size 4) admitting a refinement into six duads (called duads of F) such that the union of any three duads coming from any two tetrads is a hexad. Let

$$P_F = \big\{\{\{x,u,v\},\{y,u,v\}\} \bigm| \{x,y,u,v\} \in F,\ \{u,v\},\ \{x,y\} \text{ are duads of } F\big\}$$

and let $\mathcal{P} = \{P_F \mid F \text{ a linked four}\}$. Then $P_F \cong 6K_2$ with one copy of $2K_2$ for each tetrad in F. Let $\{A, B\}$ be an edge of $J(12,3)$. It turns out (MAGMA calculation [3]) there is exactly one linked four F having $A \cup B$ as a tetrad and $A \cap B$ as a duad of F, and so P_F is the only part of $\mathcal{P}$ containing $\{A, B\}$. Since G acts primitively on the set of linked fours, it follows that $(J(12,3), \mathcal{P})$ is an M_{12}-primitive decomposition.

Proposition 6.6. *If $(J(12,3), \mathcal{P})$ is an M_{12}-primitive decomposition then $\mathcal{P} = \mathcal{P}_\cup, \mathcal{P}_\cap$ or $\mathcal{P}_\ominus$ or $\mathcal{P}$ is obtained from Construction* 6.5.

Proof. Let $\Gamma = J(12,3)$ and $G = M_{12}$ acting on the point set X of the Witt-design $S(5,6,12)$. Take adjacent vertices $A = \{1,2,3\}$ and $B = \{2,3,4\}$. The stabiliser in G of a 4-set is $M_8 \rtimes S_4$ such that the pointwise stabiliser M_8 of the 4-set acts regularly on the 8 remaining points. Hence $G_{\{A,B\}} = G_{\{1,4\},\{2,3\}} = M_8 \rtimes (S_2 \times S_2)$ which has order 32 and is transitive on the 8 points of $X \backslash \{1,2,3,4\}$.

Let H be a maximal subgroup of G such that $G_{\{A,B\}} \leqslant H < G$. The maximal subgroups of G are given in [10], and comparing orders we see that $H \ncong M_{11}$, PSL(2, 11), $M_9 \rtimes S_3$, $C_2 \times S_5$ and $A_4 \times S_3$. Moreover, since $G_{\{A,B\}}$ has orbits of size 2, 2 and 8 in X it does not stabilise a hexad pair. If H is intransitive then $H = G_{\{1,2,3,4\}}$, $G_{\{1,4\}}$ or $G_{\{2,3\}}$. These yield $\mathcal{P}_\cup$, $\mathcal{P}_\ominus$ and $\mathcal{P}_\cap$ respectively. Thus by [10, p. 33] we are left to consider the case where $H \cong 4^2 \rtimes D_{12}$. A MAGMA [3] calculation shows that there is a unique such H containing $G_{\{A,B\}}$ and we obtain the decomposition from Construction 6.5. $\square$

Before dealing with $G = M_{11}$ we need the following couple of lemmas, the first of which is well known.

Lemma 6.7. *Let $(X, \mathcal{B})$ be the Witt design $S(4,5,11)$. Then $\mathcal{B}$ contains* 66 *elements, called* pentads. *Each point of X is contained in* 30 *pentads, each* 2*-subset in* 12 *pentads, each* 3*-subset in* 4 *pentads, and each* 4*-subset in a unique pentad. Moreover, the stabiliser of a pentad is isomorphic to S_5, which acts in its natural action on the pentad and as* PGL(2, 5) *on the complementary hexad.*

Proof. Since $(X, \mathcal{B})$ can be derived from the set of blocks of the Witt design $S(5,6,12)$ containing a given point, the first part follows from Lemma 5.1. By [10, p. 18], the stabiliser of a pentad is S_5 and has two orbits on X. $\square$

Lemma 6.8. *Let $(X, \mathcal{B})$ be the Witt design $S(4,5,11)$ and $G = M_{11}$. Let $A = \{1,2,3\}$, $B = \{2,3,4\}$ and suppose that $p = \{1,2,3,4,5\}$ is the unique pentad containing $A \cup B$. Then $G_{\{A,B\}} \cong C_2^2$ and on $X \backslash p$ has an orbit $\{a, b\}$ of length* 2 *and an orbit of length* 4. *Moreover, $\{1,4,5,a,b\}$, $\{2,3,5,a,b\}$ and $X \backslash \{1,2,3,4,a,b\}$ are pentads.*

Proof. By Lemma 6.7, G_p induces S_5 on p, and since $G_{\{A,B\}} \leqslant G_p$ it follows that $G_{\{A,B\}} = G_{\{2,3\},\{1,4\}} \cong C_2^2$ and fixes the point 5. By [10], each involution of G fixes precisely three points of X. Two of the involutions of $G_{\{A,B\}}$ fix three points of p and so are fixed point free on $X \backslash p$. The third involution fixes the point 5 and fixes two points a, b of $X \backslash p$. It follows that $G_{\{A,B\}}$ has an orbit of length two (namely, $\{a, b\}$) and an orbit of length 4 on $X \backslash p$.

Any four points lie in a unique pentad and by Lemma 6.7, any 3-subset is contained in 4 pentads. Hence $X \backslash p$ is divided into three sets of size two by the three pentads containing $\{1,4,5\}$

other than $\{1,2,3,4,5\}$. Similarly, $X\backslash p$ is partitioned by the three pentads containing $\{2,3,5\}$. Since $G_{\{A,B\}}$ fixes $\{1,4,5\}$ and $\{2,3,5\}$, it preserves both partitions and $\{a,b\}$ must be a block of both. Hence $\{1,4,5,a,b\}$ and $\{2,3,5,a,b\}$ are pentads. Moreover, since $X\backslash(\{a,b\}\cup p)$ is an orbit of length 4 of $G_{\{A,B\}}$ and is contained in a unique pentad, the fifth point of this pentad must also be fixed by $G_{\{A,B\}}$ and hence is 5. Thus $X\backslash\{1,2,3,4,a,b\}$ is a pentad. $\square$

Since the stabiliser of a 2-set is maximal in M_{11}, it follows from Lemma 2.9 that $\mathcal{P}_{\cap}$ and $\mathcal{P}_{\ominus}$ are M_{11}-primitive decompositions. We also obtain M_{11}-primitive decompositions from Constructions 2.10, 2.12, 2.14 and 2.16 by using the Witt design $S(4,5,11)$, since the stabilisers of a block, of a point and of a 3-subset are maximal subgroups of M_{11}.

Construction 6.9. Let $(X,\mathcal{B})$ be the Witt design $S(4,5,11)$ and $G=M_{11}$. Let $A=\{1,2,3\}$ and $B=\{2,3,4\}$ be adjacent vertices of $J(11,3)$ and let $\{a,b\}$ be the orbit of length 2 of $G_{\{A,B\}}$ on $X\backslash\{1,2,3,4,5\}$ given by Lemma 6.8.

(1) For each 3-subset Y of X let

$$P_Y=\bigl\{\bigl\{\{x,u,v\},\{y,u,v\}\bigr\}\bigm|\{x,y\}\cup Y,\ \{u,v\}\cup Y\in\mathcal{B}\bigr\}$$

and let $\mathcal{P}=\{P_Y \mid Y \text{ a 3-subset}\}$. By Lemma 6.7, Y is contained in 4 pentads, and so $P_Y\cong 12K_2$. Let $Y=\{5,a,b\}$. By Lemma 6.8, $\{A,B\}\in P_Y$ and $G_{\{A,B\}}\leqslant G_Y=G_{P_Y}$, which is a maximal subgroup of G. Hence by Lemma 2.4, $(J(11,3),\mathcal{P})$ is an M_{11}-primitive decomposition.

(2) Since G is 4-transitive on X, Lemma 6.8 implies that the stabiliser in G of two 2-subsets of X fixes a third. For each 2-subset Y let

$$P_Y=\bigl\{\bigl\{\{x,u,v\},\{y,u,v\}\bigr\}\bigm|u,v,x,y\in X\backslash Y,\ G_{Y,\{x,y\}}=G_{Y,\{u,v\}}\bigr\}$$

and let $\mathcal{P}=\{P_Y \mid Y \text{ a 2-subset}\}$. Then each $P_Y\cong\binom{9}{2}K_2$. Moreover, by Lemma 6.8 any edge of $J(11,3)$ is contained in a unique part of $\mathcal{P}$ ($\{A,B\}\in P_{\{a,b\}}$) and so $(J(11,3),\mathcal{P})$ is an M_{11}-primitive decomposition.

(3) For each $Y\in\mathcal{B}$ let

$$P_Y=\bigl\{\bigl\{\{x,u,v\},\{y,u,v\}\bigr\}\bigm|x,y\in Y,\ \{u,v\}\cup\bigl(Y\backslash\{x,y\}\bigr)\in\mathcal{B}\bigr\}$$

and let $\mathcal{P}=\{P_Y \mid Y\in\mathcal{B}\}$. By Lemma 6.7, each 3-subset of Y is contained in three more pentads and so each part of $\mathcal{P}$ is isomorphic to $3\binom{5}{2}K_2=30K_2$. By Lemma 6.8, $\{A,B\}\in P_Y$ for $Y=\{1,4,5,a,b\}$. Moreover, $G_{\{A,B\}}$ fixes Y and so $G_{\{A,B\}}<G_Y=G_{P_Y}$. Thus Lemma 2.4 and the fact that G acts primitively on $\mathcal{B}$, imply that $(J(11,3),\mathcal{P})$ is a G-primitive decomposition.

(4) For each $Y\in\mathcal{B}$ let

$$P_Y=\bigl\{\bigl\{\{x,u,v\},\{y,u,v\}\bigr\}\bigm|u,v\in Y,\ \{x,y\}\cup\bigl(Y\backslash\{u,v\}\bigr)\in\mathcal{B}\bigr\}$$

and let $\mathcal{P}=\{P_Y \mid Y\in\mathcal{B}\}$. By Lemma 6.7, each 3-subset of Y is contained in three more pentads and so each part of $\mathcal{P}$ is isomorphic to $3\binom{5}{2}K_2=30K_2$. By Lemma 6.8, $\{A,B\}\in P_Y$ for $Y=\{2,3,5,a,b\}$ and $G_{\{A,B\}}<G_Y=G_{P_Y}$. Thus Lemma 2.4 and the fact that G acts primitively on $\mathcal{B}$, imply that $(J(11,3),\mathcal{P})$ is a G-primitive decomposition.

Construction 6.10. Let $H=\mathrm{PSL}(2,11)<M_{11}=G$. Then H has an orbit of length 55 on 3-subsets and this orbit forms a $2-(11,3,3)$ design known as the Petersen design. The remaining 3-subsets form an orbit of length 110 and a $2-(11,3,6)$ design [5].

Table 6
M_{11}-primitive decompositions of $J(11,3)$

$\mathcal{P}$	P	G_P
$\mathcal{P}_\cap$	K_9	$M_9 \rtimes C_2$
$\mathcal{P}_\ominus$	$\binom{9}{2}K_2$	$M_9 \rtimes C_2$
Construction 2.10	$J(5,3) \cong J(5,2)$	S_5
Construction 2.12	$30K_4$	M_{10}
Construction 2.14	$4K_3$	$M_8 \rtimes S_3$
Construction 2.16	$12K_2$	$M_8 \rtimes S_3$
Construction 6.9(1)	$12K_2$	$M_8 \rtimes S_3$
Construction 6.9(2)	$\binom{9}{2}K_2$	$M_9 \rtimes C_2$
Construction 6.9(3)	$30K_2$	S_5
Construction 6.9(4)	$30K_2$	S_5
Construction 6.10(1)	Π	PSL(2, 11)
Construction 6.10(2)	11 Petersen graphs	PSL(2, 11)
Construction 6.11	2 Petersen graphs	S_5

(1) Let Π be the subgraph of $J(11,3)$ induced on the orbit of length 55. The graph Π was studied in [13] and is H-arc-transitive of valency 6. Given an edge $\{A,B\}$ of Π we have $H_{\{A,B\}} = C_2^2 = G_{\{A,B\}}$. Thus letting $\mathcal{P} = \{\Pi^g \mid g \in G\}$, it follows by Lemma 2.4 that $(J(11,3),\mathcal{P})$ is a G-primitive decomposition.

(2) Let Δ be the subgraph of $J(11,3)$ induced on the orbit of length 110. Then Δ has valency 15 and given a vertex A, $H_A \cong S_3$ has orbits of length 3, 6 and 6 on the neighbours of A. Let B be a neighbour of A in the orbit of length 3 and let $P = \{A,B\}^H$. Let $g \in H$ which interchanges A and B. Then by Lemma 2.18, $P \cong \mathrm{Cos}(H, H_A, H_A g H_A)$. Moreover, $\langle H_A, g\rangle \cong A_5$ and so P has 11 connected components, each with 10 vertices and isomorphic to the Petersen graph. Since $|H_{\{A,B\}}| = 4 = |G_{\{A,B\}}|$, it follows from Lemma 2.4 that $(J(11,3),\mathcal{P})$ is a G-primitive decomposition with $\mathcal{P} = P^G$.

Construction 6.11. Let $A = \{1,2,3\}$ and $B = \{2,3,4\}$. By Lemma 6.8, $Y = X\backslash\{1,2,3,4,a,b\}$ is a pentad fixed by $G_{\{A,B\}}$. Let $H = G_Y$ and $P = \{A,B\}^H$. Then by Lemma 6.7, H induces S_5 on Y and $\mathrm{PGL}(2,5)$ on $\{1,2,3,4,a,b\}$. Thus $H_A \cong S_3$ and is a maximal subgroup of $A_5 \cong \mathrm{PSL}(2,5)$. Moreover, $g \in H_{\{A,B\}}$ which interchanges A and B induces even permutations on Y and so for such a g we have $\langle H_A, g\rangle = A_5$. By Lemma 2.18, $P \cong \mathrm{Cos}(H, H_A, H_A h H_A)$. Since $|H : H_A| = 20$ and $\langle H_A, g\rangle \cong A_5$, it follows that P has two disconnected components with 10 vertices each. Since $|H_A : G_{A,B}| = 3$ it follows that P is a copy of two Petersen graphs. Let $\mathcal{P} = P^G$. Then as $G_{\{A,B\}} < H$, it follows from Lemma 2.4 that $(J(11,3),\mathcal{P})$ is a G-primitive decomposition.

Proposition 6.12. *If $(J(11,3),\mathcal{P})$ is an M_{11}-primitive symmetric decomposition then $\mathcal{P}$ is given by Table* 6.

Proof. Let $\Gamma = J(11,3)$ and $G = M_{11} < \mathrm{Sym}(X)$, and consider X as the point set of the Witt-design $S(4,5,11)$ with automorphism group G. Let $A = \{1,2,3\}$ and $B = \{2,3,4\}$ be adjacent vertices. Suppose that $p = \{1,2,3,4,5\}$ is the unique pentad of the Witt design containing $\{1,2,3,4\}$ and let H be a maximal subgroup of G containing $G_{\{A,B\}} = G_{\{2,3\},\{1,4\}}$. The maximal subgroups of G are given in [10, p. 18].

If H is the stabiliser of a point then $H = G_5$ and so we obtain the decomposition yielded by Construction 2.12. Next suppose that H is the stabiliser of a duad. Then H is one of $G_{\{2,3\}}$, $G_{\{1,4\}}$ or $G_{\{a,b\}}$ where $\{a, b\}$ is the orbit of length two of $G_{\{A,B\}}$ on $\{6, 7, \dots, 11\}$. The first gives $\mathcal{P}_\cap$ the second gives $\mathcal{P}_\ominus$. Finally, if $H = G_{\{a,b\}}$ then H is the stabiliser of the divisor of the decomposition obtained from Construction 6.9(2) containing $\{A, B\}$.

Next suppose that H is the stabiliser of a triad. Then H stabilises $\{1, 4, 5\}$, $\{2, 3, 5\}$ or $\{5, a, b\}$. If $H = G_{\{1,4,5\}}$ then H is the stabiliser of the divisor of the decomposition from Construction 2.14 containing $\{A, B\}$. Also $H = G_{\{2,3,5\}}$ is the stabiliser of the divisor of the decomposition yielded by Construction 2.16 containing $\{A, B\}$. Finally, $H = G_{\{5,a,b\}}$ is the stabiliser of the divisor of the decomposition obtained from Construction 6.9(1) containing $\{A, B\}$.

Next suppose that H is the stabiliser of a pentad. Since $G_{\{A,B\}}$ has only one orbit of odd length, it follows that 5 is in the pentad. Combining 5 with two orbits of $G_{\{A,B\}}$ of length two we get that $G_{\{A,B\}}$ fixes the pentads $\{1, 2, 3, 4, 5\}$, $\{1, 4, 5, a, b\}$, $\{2, 3, 5, a, b\}$ and $X\backslash\{1, 2, 3, 4, a, b\}$ (by Lemma 6.8, these 5-sets are actually pentads). Thus there are four choices for H. If $H = G_{\{1,2,3,4,5\}}$ then we obtain the decomposition from Construction 2.10. If $H = G_{\{1,4,5,a,b\}}$, then H is the stabiliser of the divisor of the decomposition from Construction 6.9(3) containing $\{A, B\}$ while $H = G_{\{2,3,5,a,b\}}$ is the stabiliser of the divisor of the decomposition yielded by Construction 6.9(4). Finally, if $H = G_{X\backslash\{1,2,3,4,a,b\}}$ then H is the stabiliser of the divisor of the decomposition produced by Construction 6.11 containing $\{A, B\}$.

We are left to consider $H \cong \mathrm{PSL}(2, 11)$. By a calculation using MAGMA [3], there are two such H containing $G_{\{A,B\}}$. These give us the two decompositions in Construction 6.10. □

We now give constructions for $\mathrm{P\Gamma L}(2, 8)$-primitive decompositions of $J(9, 3)$.

Construction 6.13. Let $G = \mathrm{P\Gamma L}(2, 8)$ and $X = \mathrm{GF}(8) \cup \{\infty\}$, where GF(8) is defined by the relation $i^3 = i + 1$.

(1) By Theorem 3.4, $T = \mathrm{PSL}(2, 8)$ is not arc-transitive on $J(9, 3)$ and so as $T \lhd G$ and has index three, T has three equal sized orbits on edges. Thus the partition $\mathcal{P} = \{P_1, P_2, P_3\}$ given by these three orbits is a G-primitive decomposition. Since T is vertex-transitive, this is in fact a homogeneous factorisation and appears in [11].

(2) Let $x \in X$. Then $G_x = \mathrm{A\Gamma L}(1, 8)$ and preserves the structure of an affine space AG(3, 2) (with plane-set $\mathcal{B}$) on $X\backslash\{x\}$. Let

$$P_x = \bigl\{\{A, B\} \bigm| A \cup B \in \mathcal{B}\bigr\}$$

and $\mathcal{P} = \{P_x \mid x \in X\}$. Then since each 3-subset lies in a unique plane, $P_x \cong 14K_4$. Moreover, G_x acts transitively on the set $\mathcal{B}$ of affine planes and for $Y \in \mathcal{B}$ we have $G_{x,Y}$ induces A_4 on Y. Thus G_x acts transitively on the set of edges in P_x and so given $\{A, B\} \in P_x$ we have $|G_{x,\{A,B\}}| = 2 = |G_{\{A,B\}}|$. Thus $G_{\{A,B\}} \leqslant G_x$ and so by Lemma 2.4, $\mathcal{P} = P_x^G$ is a G-primitive decomposition of $J(9, 3)$.

(3) Let $A = \{\infty, 0, 1\}$ and $B = \{\infty, 0, i\}$. Then $G_{\{A,B\}} = \langle g\rangle \cong C_2$ where $x^g = ix^{-1}$ and has orbits $\{0, \infty\}$, $\{1, i\}$, $\{i^2, i^6\}$, $\{i^3, i^5\}$ and $\{i^4\}$. Thus $G_{\{A,B\}} \leqslant G_{\{i^2,i^6\}} = H$ (H has order 42) and so by Lemma 2.4, letting $P = \{A, B\}^H$ and $\mathcal{P} = P^G$ we obtain a G-primitive decomposition of $J(9, 3)$. Now $H_A = \langle h\rangle$ where $x^h = x + 1$, which has order two and so P has 21 vertices and valency 2. Moreover, $\langle H_A, g\rangle = D_{14}$ and so by Lemma 2.18, P has three connected components. Thus $P \cong 3C_7$.

Table 7
$\mathrm{P\Gamma L}(2,8)$-primitive decompositions of $J(9,3)$

$\mathcal{P}$	P	G_P
$\mathcal{P}_\cap$	K_7	$D_{14} \rtimes C_3$
$\mathcal{P}_\ominus$	$\binom{7}{2}K_2$	$D_{14} \rtimes C_3$
Construction 6.13(1)	PSL(2, 8)-orbits	PSL(2, 8)
Construction 6.13(2)	$14K_4$	$\mathrm{A\Gamma L}(1,8)$
Construction 6.13(3)	$3C_7$	$D_{14} \rtimes C_3$
Construction 6.13(4)	Heawood graph	$D_{14} \rtimes C_3$
Construction 6.14(1)	$3C_9$	$D_{18} \rtimes C_3$
Construction 6.14(2)	$27K_2$	$D_{18} \rtimes C_3$
Construction 6.14(3)	$27K_2$	$D_{18} \rtimes C_3$
Construction 6.14(4)	$27K_2$	$D_{18} \rtimes C_3$

(4) Let $A = \{\infty, 0, 1\}$ and $B = \{\infty, 0, i\}$. Then $G_{\{A,B\}} \leqslant G_{\{i^3,i^5\}} = H$ and so by Lemma 2.4, letting $P = \{A, B\}^H$ and $\mathcal{P} = P^G$ we obtain a G-primitive decomposition of $J(9,3)$. Then $H_A = \langle h\rangle$ where $x^h = (x^4+1)^{-1}$, which has order three. Thus P has 14 vertices and valency 3. Since g and h do not commute, $\langle H_A, g\rangle = H$ and so P is a connected graph. Moreover, P is H-arc-transitive and so by [31, p. 167], P is the Heawood graph.

Construction 6.14. Let $K = \mathrm{GF}(64)$, with ξ a primitive element of K, and let $F = \{0\} \cup \{(\xi^9)^l \mid l = 0, 1, \ldots, 6\} \cong \mathrm{GF}(8)$. One can consider the projective line X on which G acts as the elements of K modulo F. Then $H = \langle \hat{\xi}, \sigma, \tau\rangle \cong D_{18} \rtimes C_3$ where $\hat{\xi} : x \to \xi x \pmod F$, $\sigma : x \to x^8 = x^{-1} \pmod F$, and $\tau : x \to x^4 \pmod F$.

(1) Let $A = \{1, \xi, \xi^2\}$ and $B = \{\xi, \xi^2, \xi^3\}$. Then $\{A, B\}$ is an edge of $J(9,3)$ whose ends are interchanged by $\hat{\xi}^6\sigma \in H$. Thus letting $P = \{A, B\}^H$ and $\mathcal{P} = P^G$, Lemma 2.4 implies that $(J(9,3), \mathcal{P})$ is a G-primitive decomposition. Now $H_A = \langle \hat{\xi}^7\sigma\rangle$ and so P has 27 vertices. Moreover, $H_{A,B} = 1$ and so P has valency 2. Since $\langle \hat{\xi}^6\sigma, \hat{\xi}^7\sigma\rangle = D_{18}$ it follows from Lemma 2.18 that P has 3 connected components and so $P \cong 3C_9$.
(2) Let $A = \{1, \xi, \xi^3\}$ and $B = \{1, \xi, \xi^7\}$. Then $\{A, B\}$ is an edge of $J(9,3)$ whose ends are interchanged by $\hat{\xi}^8\sigma \in H$. Thus letting $P = \{A, B\}^H$ and $\mathcal{P} = P^G$, Lemma 2.4 implies that $(J(9,3), \mathcal{P})$ is a G-primitive decomposition. Now $|H_A| = 1$ and so P is a matching of 27 edges.
(3) Let $A = \{1, \xi, \xi^3\}$ and $B = \{\xi, \xi^3, \xi^4\}$. Then $\{A, B\}$ is an edge of $J(9,3)$ whose ends are interchanged by $\hat{\xi}^5\sigma \in H$. Thus letting $P = \{A, B\}^H$ and $\mathcal{P} = P^G$, Lemma 2.4 implies that $(J(9,3), \mathcal{P})$ is a G-primitive decomposition. Now $|H_A| = 1$ and so P is a matching of 27 edges.
(4) Let $A = \{1, \xi, \xi^3\}$ and $B = \{1, \xi^2, \xi^3\}$. Then $\{A, B\}$ is an edge of $J(9,3)$ whose ends are interchanged by $\hat{\xi}^6\sigma \in H$. Thus letting $P = \{A, B\}^H$ and $\mathcal{P} = P^G$, Lemma 2.4 implies that $(J(9,3), \mathcal{P})$ is a G-primitive decomposition. Now $|H_A| = 1$ and so P is a matching of 27 edges.

Proposition 6.15. *If $(J(9,3), \mathcal{P})$ is a $\mathrm{P\Gamma L}(2,8)$-primitive decomposition then $\mathcal{P}$ is as in Table* 7.

Proof. Let $G = \mathrm{P\Gamma L}(2,8)$ act on $\{\infty\} \cup \mathrm{GF}(8)$ and suppose that GF(8) has primitive element i such that $i^3 = i + 1$. Let $A = \{\infty, 0, 1\}$ and $B = \{\infty, 0, i\}$ be adjacent vertices in $\Gamma = J(9,3)$.

Then $G_{\{A,B\}} = G_{\{\infty,0\},\{1,i\}} = \langle g\rangle \cong C_2$, where $x^g = ix^{-1}$, which fixes the point i^4 and has 4 orbits of size 2. Let H be a maximal subgroup of G containing $G_{\{A,B\}}$. The maximal subgroups of G are given in [10, p. 6].

If $H = \mathrm{PGL}(2,8)$ then we obtain the decomposition in Construction 6.13(1) while if H is a point stabiliser then $H = G_{i^4}$ and we obtain the decomposition in Construction 6.13(2).

Suppose now that $H \cong D_{14} \rtimes C_3$ is the stabiliser of a 2-subset. Then $H = G_{\{\infty,0\}}$, $H = G_{\{1,i\}}$, $H = G_{\{i^2,i^6\}}$, or $H = G_{\{i^3,i^5\}}$. In the first case we get the decomposition $\mathcal{P}_\cap$, while the second yields $\mathcal{P}_\ominus$. The third case gives Construction 6.13(3) and the fourth gives the decomposition in Construction 6.13(4).

Let $H = \langle\hat{\xi},\sigma,\tau\rangle \cong D_{18} \rtimes C_3$ as given in Construction 6.14. Instead of finding all conjugates of H containing $G_{\{A,B\}}$, we (equivalently) find all edge orbits $\{C,D\}^H$ such that H contains $G_{\{C,D\}}$. Note that, for such an edge, C and D lie in the same H-orbit on vertices. One sees easily that H has three orbits on vertices of $J(9,3)$, of sizes 3 ($\{1,\xi^3,\xi^6\}^{\langle\hat{\xi}\rangle}$), 27 ($\{1,\xi,\xi^2\}^{\langle\hat{\xi}\rangle} \cup \{1,\xi^2,\xi^4\}^{\langle\hat{\xi}\rangle} \cup \{1,\xi^4,\xi^8\}^{\langle\hat{\xi}\rangle}$), and 54 (all the other vertices). The orbit of size 3 contains no edges. In the orbit of size 27, if we fix the vertex $C = \{1,\xi,\xi^2\}$, we find two vertices D, namely $\{1,\xi,\xi^8\}$ and $\{\xi,\xi^2,\xi^3\}$, such that the unique involution switching C and D is in H. Moreover, these two vertices are interchanged by H_C. Hence this vertex orbit yields one orbit of edges whose stabilisers are contained in H and we get the decomposition in Construction 6.14(1).

In the orbit of size 54, if we fix the vertex $C = \{1,\xi,\xi^3\}$, we find three vertices D, namely $\{1,\xi,\xi^7\}$, $\{\xi,\xi^3,\xi^4\}$ and $\{1,\xi^2,\xi^3\}$, such that the unique involution switching C and D is in H. Since H acts regularly on this orbit, each choice of D gives a different H-orbit on edges and we get the three decompositions of Constructions 6.14(2)–(4). □

7. The case $k = 2$

By Theorem 3.4, a subgroup G of S_n is arc-transitive on $J(n,2)$ if and only if G is 3-transitive. Hence other than A_n or S_n, the possibilities for (n,G) are $(11,M_{11})$, $(12,M_{11})$, $(12,M_{12})$, $(22,M_{22})$, $(22,\mathrm{Aut}(M_{22}))$, $(23,M_{23})$, $(24,M_{24})$, $(2^d,\mathrm{AGL}(d,2))$ for $d > 2$, $(16, C_2^4 \rtimes A_7)$, and $(q+1,G)$ where G is a 3-transitive subgroup of $\mathrm{P\Gamma L}(2,q)$ with $q \geqslant 4$. We treat all but the last case in this section and deal with the subgroups of $\Gamma\mathrm{L}(2,q)$ in Section 8.

Proposition 7.1. *If $(J(11,2),\mathcal{P})$ is an M_{11}-primitive decomposition then $\mathcal{P}$ is $\mathcal{P}_\cap$, $\mathcal{P}_\cup$, or $\mathcal{P}_\ominus$.*

Proof. Let $G = M_{11}$ act on the point set X of the Witt design $S(4,5,11)$, and let $A = \{1,2\}$, $B = \{2,3\}$ be adjacent vertices. Then $G_{\{A,B\}} = G_{2,\{1,3\}}$ and since G is strictly 4-transitive it follows that $|G_{\{A,B\}}| = 16$ and has one orbit on the 8 remaining points. Let H be a maximal subgroup of G containing $G_{\{A,B\}}$. Comparing orders and the maximal subgroups of G given in [10, p. 18] we see that $H \ncong \mathrm{PSL}(2,11)$ or S_5. It follows that H stabilises either a point, a pair or a 3-subset. In the first case $H = G_2$ and so $\mathcal{P} = \mathcal{P}_\cap$. In the second case, $H = G_{\{1,3\}}$ and we obtain the decomposition $\mathcal{P}_\ominus$, while in the last case $H = G_{\{1,2,3\}}$ and so we get the decomposition $\mathcal{P}_\cup$. □

Since the stabilisers of a point and a 2-subset of a 12-set are maximal in M_{11} it follows from Lemma 2.9 that $\mathcal{P}_\cap$ and $\mathcal{P}_\ominus$ are M_{11}-primitive decompositions of $J(12,2)$. In order to give more constructions for M_{11}-primitive decompositions of $J(12,2)$, we will need the following lemma.

Lemma 7.2. *Let $G = M_{11}$ act 3-transitively on the point set X of the Witt design $S(5,6,12)$. As seen in Construction 5.6, G has an orbit of length 165 on 4-subsets, forming a $3-(12,4,3)$ design with block set $\mathcal{D}$. In this design, each 3-set S determines uniquely another 3-set $S_{\mathcal{D}}$, namely the set of fourth points of the 3 blocks of $\mathcal{D}$ containing S. We have $(S_{\mathcal{D}})_{\mathcal{D}} = S$ and $S \cup S_{\mathcal{D}}$ is a hexad of $S(5,6,12)$. Moreover if $\{S, S_{\mathcal{D}}, U, V\}$ is the unique linked three containing S and $S_{\mathcal{D}}$ as triads (see Lemma 5.4), then $U_{\mathcal{D}} = V$.*

Proof. For any 3-set S, the set $S_{\mathcal{D}}$ is obviously well defined by the properties of the $3-(12,4,3)$ design. Now, an element of G stabilising S must also stabilise $S_{\mathcal{D}}$. Therefore $G_S \leqslant G_{S_{\mathcal{D}}}$. Since $S_{\mathcal{D}}$ is also a 3-set and G is 3-transitive, we must have $|G_S| = |G_{S_{\mathcal{D}}}|$. Therefore $G_S = G_{S_{\mathcal{D}}}$. By a computation using MAGMA [3] we find that $G_S \cong S_3 \times S_3$ has orbits of lengths 3, 3 and 6 on X. Hence $(S_{\mathcal{D}})_{\mathcal{D}} = S$.

Let u, v be two points of $S_{\mathcal{D}}$. Then $S \cup \{u, v\}$ is contained in a unique hexad h. Since G_S preserves the set of hexads containing S, and acts transitively on the 3 points of $S_{\mathcal{D}}$ and on the 6 points of $X \backslash (S \cup S_{\mathcal{D}})$, it follows that the sixth point of h must also lie in $S_{\mathcal{D}}$. Hence $S \cup S_{\mathcal{D}}$ is a hexad. Let $T = \{S, S_{\mathcal{D}}, U, V\}$ be the unique linked three containing S and $S_{\mathcal{D}}$ as triads (Lemma 5.4). Since G_S preserves T and is transitive on $U \cup V$, it follows that G_S has an index 2 subgroup $G_{S,U}$ with orbits $S, S_{\mathcal{D}}, U$ and V. Since the orbits of $G_{S,U}$ are a refinement of the orbits of G_U, $U_{\mathcal{D}}$ must be one of these orbits of size 3. Since $U_{\mathcal{D}}$ cannot be S nor $S_{\mathcal{D}}$, it follows that $U_{\mathcal{D}} = V$. □

Construction 7.3. Let $G = M_{11}$ act 3-transitively on the point set X of the Witt design $S(5,6,12)$. We use the notation of Lemma 7.2.

(1) Let $Y \in \mathcal{D}$. Let

$$P_Y = \big\{ \big\{ \{u,x\}, \{x,v\} \big\} \bigm| \{x,u,v\}_{\mathcal{D}} = Y \backslash \{x\} \big\}$$

and $\mathcal{P} = \{P_Y \mid Y \in \mathcal{D}\}$. Then $P_Y \cong 4K_2$. Let $\{\{u,x\},\{x,v\}\}$ be an edge of $J(12,2)$. Then it is in a unique P_Y, with $Y = \{x\} \cup \{x,u,v\}_{\mathcal{D}}$. Since G_Y is maximal in G, it follows that $(J(12,2), \mathcal{P})$ is a G-primitive decomposition.

(2) Let T be a $\mathcal{D}$-linked three, that is, a linked three for the $S(5,6,12)$ such that, for any $X \in T$, $X_{\mathcal{D}}$ is a triad of T. Let

$$P_T = \big\{ \big\{ \{u,x\}, \{x,v\} \big\} \bigm| \{x,u,v\} \in T \big\}$$

and $\mathcal{P} = \{P_T \mid T \text{ is a } \mathcal{D}\text{-linked three}\}$. Then $P_T \cong 4K_3$, with each triad contributing K_3. Let $\{\{u,x\},\{x,v\}\}$ be an edge of $J(12,2)$. Then $\{u,v,x\}$ and $\{u,v,x\}_{\mathcal{D}}$ must be triads of T. By Lemma 7.2, the unique linked three containing these two triads is a $\mathcal{D}$-linked three. It follows that there is exactly one $\mathcal{D}$-linked three T such that P_T contains a given edge. Since the stabiliser in G of a $\mathcal{D}$-linked three is maximal in G, it follows that $(J(12,2), \mathcal{P})$ is a G-primitive decomposition.

Thus we have the M_{11}-primitive decompositions listed in Table 8.

Proposition 7.4. *If $(J(12,2), \mathcal{P})$ is an M_{11}-primitive decomposition then $\mathcal{P}$ is given by Table 8.*

Proof. Let $G = M_{11}$ act transitively on the point set X of the Witt design $S(5,6,12)$ and let $\mathcal{D}$ be the block set of the $3-(12,4,3)$ design described in Construction 5.6 (see above). Take adjacent

Table 8
M_{11}-primitive decompositions of $J(12,2)$

$\mathcal{P}$	P	G_P
$\mathcal{P}_\cap$	K_{11}	PSL(2, 11)
$\mathcal{P}_\ominus$	$10K_2$	S_5
Construction 7.3(1)	$4K_2$	$M_8 \rtimes S_3$
Construction 7.3(2)	$4K_3$	$M_9 \rtimes C_2$

vertices $A=\{1,2\}$ and $B=\{2,3\}$. Then $G_{\{A,B\}}=G_{2,\{1,3\}}\cong D_{12}$ which has an orbit of length 3 (namely, $\{1,2,3\}_{\mathcal{D}}$) and an orbit of length 6 on the remaining 9 points of X. Let H be a maximal subgroup of G containing $G_{\{A,B\}}$. Since M_{10} contains no elements of order 6, it follows that $H\ncong M_{10}$. If H is a point stabiliser, then $H=G_2$ and we get the decomposition $\mathcal{P}_\cap$. If H is a pair stabiliser then $H=G_{\{1,3\}}$, and we get the decomposition $\mathcal{P}_\ominus$. If $H\cong M_8\rtimes S_3$ then H is the stabiliser of a block in $\mathcal{D}$. There is a unique such block, namely the union of $\{2\}$ with $\{1,2,3\}_{\mathcal{D}}$. Hence H is the stabiliser of the divisor of the decomposition obtained from Construction 7.3(1) containing $\{A,B\}$.

Now let $H\cong M_9\rtimes C_2$. Then H is a $\mathcal{D}$-linked three stabiliser, namely the only one containing $\{1,2,3\}$ as a triad (see the construction). Hence H is the stabiliser of the divisor of the decomposition obtained from Construction 7.3(2) containing $\{A,B\}$. □

Proposition 7.5. *If $(J(12,2),\mathcal{P})$ is an M_{12}-primitive decomposition, then $\mathcal{P}$ is $\mathcal{P}_\cup$, $\mathcal{P}_\cap$ or $\mathcal{P}_\ominus$.*

Proof. Let $G=M_{12}$ act on the point set X of the Witt-design $S(5,6,12)$ and take adjacent vertices $A=\{1,2\}$ and $B=\{2,3\}$. Then $G_{\{A,B\}}=G_{2,\{1,3\}}$ which has order 144 and is 2-transitive on the 9 remaining points since G is 5-transitive on X. Let H be a maximal subgroup of G containing $G_{\{A,B\}}$. The maximal subgroups of G are given in [10], and comparing orders we see that $H\ncong \mathrm{PSL}(2,11)$, $2\times S_5$, $4^2:D_{12}$, $M_8.S_4$ or $A_4\times S_3$. Since $G_{\{A,B\}}$ fixes a point but not a hexad it follows that H is not the stabiliser of a hexad pair, and since $G_{\{A,B\}}$ is 2-transitive on $X\backslash\{1,2,3\}$ we also have that H is not the stabiliser of a linked three. In the action of M_{11} on 12 points, PSL(2, 11) is the stabiliser of a point. Since 144 does not divide $|\mathrm{PSL}(2,11)|$ and $G_{\{A,B\}}$ fixes the point 2, it follows that H is not a transitive copy of M_{11}. Thus $H=G_2, G_{\{1,3\}}$ or $G_{\{1,2,3\}}$. In the first case we get the decomposition $\mathcal{P}_\cap$, the second case yields $\mathcal{P}_\ominus$ while the third gives $\mathcal{P}_\cup$. □

Before dealing with $G=M_{22}$ we need the following well known result which follows from Lemma 6.3.

Lemma 7.6. *Let $(X,\mathcal{B})$ be the Witt design $S(3,6,22)$. Then $\mathcal{B}$ contains 77 elements, called* hexads. *Each point of X is contained in 21 hexads, each 2-subset in 5 hexads, and each 3-subset in a unique hexad. Moreover, the stabiliser of a hexad is $C_2^4\rtimes A_6$ with the pointwise stabiliser of the hexad being C_2^4 which acts regularly on the 16 points not in the hexad.*

Proof. Since $(X,\mathcal{B})$ can be derived from the set of blocks of the Witt design $S(4,5,23)$ containing a given point, this follows from Lemma 6.3. □

Proposition 7.7. *If $(J(22,2),\mathcal{P})$ is an M_{22}-primitive decompositions then $\mathcal{P}=\mathcal{P}_\cap$ or $\mathcal{P}_\ominus$, or $\mathcal{P}$ is obtained from Construction 2.10 and has divisors isomorphic to $J(6,2)$.*

Proof. Let $G = M_{22}$ act on the point set X of the Witt design $S(3,6,22)$ and take adjacent vertices $A = \{1,2\}$ and $B = \{2,3\}$. Moreover, suppose that $h = \{1,2,3,4,5,6\}$ is the unique hexad of the Witt design containing $\{1,2,3\}$. By Lemma 7.6, $G_h = C_2^4 \rtimes A_6$, where C_2^4 acts trivially on h and transitively on $X \backslash h$. It follows that $G_{\{A,B\}} = G_{2,\{1,3\},\{4,5,6\}}$ has order 96 and acts transitively on $X \backslash h$.

Let H be a maximal subgroup of G containing $G_{\{A,B\}}$. Comparing orders and the maximal subgroups of G given in [10] we see that $H \ncong \mathrm{PSL}(2,11)$, A_7 or M_{10}. Since $G_{\{A,B\}}$ does not stabilise an octad, it follows that H is either G_2, $G_{\{1,3\}}$ or G_h. The first gives the decomposition $\mathcal{P}_\cap$, while the second yields $\mathcal{P}_\ominus$. Finally G_h is the stabiliser of the part of the decomposition obtained from Construction 2.10 containing $\{A, B\}$ and has divisors isomorphic to $J(6,2)$. □

Proposition 7.8. *All* $\mathrm{Aut}(M_{22})$*-primitive decompositions of* $J(22,2)$ *are* M_{22}*-primitive decompositions.*

Proof. By [10], a maximal subgroup of $\mathrm{Aut}(M_{22})$ is either M_{22} or arises from a maximal subgroup of M_{22}. Since M_{22} is arc-transitive it does not give a decomposition. In all other cases, Lemma 2.7 implies that we get M_{22}-primitive decompositions. □

Proposition 7.9. *If* $(J(23,2), \mathcal{P})$ *is an* M_{23}*-primitive decomposition then* $\mathcal{P}$ *is* $\mathcal{P}_\cap$, $\mathcal{P}_\ominus$ *or* $\mathcal{P}_\cup$.

Proof. Let $G = M_{23}$ act on the point set X of the Witt design $S(4,7,23)$ and take adjacent vertices $A = \{1,2\}$ and $B = \{2,3\}$. Then $G_{\{A,B\}} = G_{2,\{1,3\}} \cong 2^4 \rtimes S_5$ (see [10, p. 71]) and since G is 4-transitive, $G_{\{A,B\}}$ is transitive on $X \backslash \{1,2,3\}$. Let H be a maximal subgroup of G containing $G_{\{A,B\}}$. Since $|G_{\{A,B\}}|$ does not divide 23.11, it follows from [10, p. 71] that H is intransitive. Hence H is G_2, $G_{\{1,3\}}$ or $G_{\{1,2,3\}}$. These give us the decompositions $\mathcal{P}_\cap$, $\mathcal{P}_\ominus$ and $\mathcal{P}_\cup$ respectively. □

Proposition 7.10. *If* $(J(24,2), \mathcal{P})$ *is an* M_{24}*-primitive symmetric decompositions then* $\mathcal{P}$ *is* $\mathcal{P}_\cap$, $\mathcal{P}_\ominus$ *or* $\mathcal{P}_\cup$.

Proof. Let $G = M_{24}$ acting on the point set X of the Witt design $S(5,8,24)$ and take adjacent vertices $A = \{1,2\}$ and $B = \{2,3\}$. Then $G_{\{A,B\}} = G_{2,\{1,3\}} \cong \mathrm{P\Sigma L}(3,4)$ (see [10, p. 96]). Note that $G_{\{A,B\}}$ is transitive on $X \backslash \{1,2,3\}$ since G is 5-transitive on X. Let H be a maximal subgroup of G containing $G_{\{A,B\}}$. Looking at the maximal subgroups of G in [10], it follows that H is either G_2, $G_{\{1,3\}}$ or $G_{\{1,2,3\}}$. Thus we obtain the decompositions $\mathcal{P}_\cap$, $\mathcal{P}_\ominus$ and $\mathcal{P}_\cup$ respectively. □

Let $G = \mathrm{AGL}(d,2)$ acting on the set X of vectors of a d-dimensional vector space over GF(2). Since the stabiliser of a vector is maximal in G, Lemma 2.9 implies that $\mathcal{P}_\cap$ is a G-primitive decomposition. The set of affine planes in the affine space $\mathrm{AG}(d,2)$ yields an $S(3,4,2^d)$ Steiner system with each point contained in $\frac{(2^d-1)(2^{d-1}-1)}{3}$ planes. Since G acts transitively on planes we can use Construction 2.10. However, G is not primitive on planes as it preserves parallelness. Applying now Construction 2.1 yields line 2 of Table 9. As G is transitive on points and the stabiliser of a point is maximal in G, applying Construction 2.12 yields line 3 of Table 9. As G is 2-transitive, we can use Construction 2.16. However, G acts imprimitively on 2-subsets as 2-subsets correspond to lines and again G preserves parallelness. Thus we also apply Construction 2.1 and obtain line 4 of Table 9. Indeed the divisors are indexed by lines of the affine

Table 9
G-primitive decompositions of $J(2^d,2)$ for $G = \mathrm{AGL}(d,2)$ with $d \geqslant 3$, or $G = C_2^4 \rtimes A_7$ with $d = 4$

$\mathcal{P}$	P	G_P
$\mathcal{P}_\cap$	K_{2^d-1}	G_0
Constructions 2.10 and 2.1	$2^{d-2}J(4,2) \cong 2^{d-2}K_{2,2,2}$	$C_2^d \rtimes (G_0)_{\langle v,w\rangle}$
Construction 2.12	$\frac{(2^d-1)(2^{d-1}-1)}{3}K_3$	G_{v+w}
Constructions 2.16 and 2.1	$2^{d-2}(2^{d-1}-1)C_4$	$C_2^d \rtimes (G_0)_{\langle v+w\rangle}$

plane and are isomorphic to $2^{d-2}K_2$. Each pair Y_1, Y_2 of parallel lines yields a C_4 in the $J(4,2)$ induced on $Y_1 \cup Y_2$. As a parallel class of lines contains 2^{d-1} lines, we have $\frac{2^{d-1}(2^{d-1}-1)}{2}$ pairs of parallel lines in the imprimitivity class.

When $d = 4$ the group $\overline{G} = C_2^4 \rtimes A_7 < \mathrm{AGL}(4,2)$ is 3-transitive on X and hence, by Corollary 3.3, is arc-transitive on $J(2^4,2)$. Thus the four G-primitive decompositions in Table 9 are also $\overline{G}$-transitive. The stabiliser in $\overline{G}$ of a point is A_7 which is maximal in $\overline{G}$. Hence the partitions in rows 1 and 3 are $\overline{G}$-primitive. The stabilisers of 2-spaces and 1-spaces in A_7 are maximal in A_7 and so the remaining two partitions are also $\overline{G}$-primitive.

Before showing that these are the only G-primitive decompositions with $G \leqslant \mathrm{AGL}(d,2)$ we need a lemma.

Lemma 7.11. *Let $G = N \rtimes G_0$ where $N \cong C_p^d$ for some prime p and G_0 acts irreducibly on N. Suppose that H is a maximal subgroup of G. Then either H is a complement of N, or $H = N \rtimes H_0$ for some maximal subgroup H_0 of G_0.*

Proof. Since H normalises N we have $H \leqslant NH \leqslant G$. Thus as H is maximal, either $NH = H$ or $NH = G$. The first case implies that $N \leqslant H$ and so $H = N \rtimes H_0$ for some maximal subgroup H_0 of G_0. Suppose now that $NH = G$. Then $H/(H \cap N) \cong G_0$, and so for each $g \in G_0$, there exists $n \in N$ such that $ng \in H$. Since N is abelian, it follows that H induces G_0 in its action on N by conjugation. Since G_0 acts irreducibly on N and H normalises $H \cap N$, it follows that $H \cap N = 1$ or N. However, $H \cap N = N$ implies that $H = G$ which is not the case. Hence $H \cap N = 1$ and $H \cong G_0$, that is H is a complement of N. □

Proposition 7.12. *Let $d \geqslant 3$ and $G = \mathrm{AGL}(d,2)$, or $d = 4$ and $G = C_2^4 \rtimes A_7$. If $(J(2^d,2), \mathcal{P})$ is a G-primitive decomposition then $\mathcal{P}$ is given by Table 9.*

Proof. We can identify X with the vectors of a d-dimensional vector space over GF(2). Let $A = \{0, v\}$ and $B = \{0, w\}$ where v, w are distinct nonzero elements of X. Thus $G_{\{A,B\}} = (G_0)_{\{v,w\}}$ which is an index 3 subgroup of $(G_0)_{\langle v,w\rangle}$. Moreover, $G_{\{A,B\}}$ fixes the vector $v + w$ and is transitive on $X \setminus \langle v, w\rangle$.

Let H be a maximal subgroup of G containing $G_{\{A,B\}}$. By Lemma 7.11, either H is a complement of $N = \mathrm{soc}(G)$ or $H = N \rtimes H_0$ for some maximal subgroup H_0 of G_0.

Suppose first that H is a complement. By a Magma [3] calculation, $C_2^4 \rtimes A_7$ has a unique conjugacy class of complements. If $d \geqslant 4$ then there is a unique class of complements of N in $\mathrm{AGL}(d,2)$, while in $\mathrm{AGL}(3,2)$ there are two classes (see for example [14]). Hence either H is the stabiliser of a vector or $d = 3$ and H is transitive. In the second case $H = \mathrm{PSL}(2,7)$ acting transitively on V. However, a Sylow 2-subgroup of H is then regular on V, and hence H cannot contain $G_{\{A,B\}} \cong D_8$ (fixing the point 0). Thus H is the stabiliser of a vector and so $H = G_0$

or G_{v+w}. The first case yields the decomposition $\mathcal{P}_\cap$, while the second is the stabiliser of the divisor of the decomposition obtained from Construction 2.12 containing $\{A, B\}$.

Suppose now that $H = N \rtimes H_0$ for some maximal subgroup H_0 of G_0. First let $G = \mathrm{AGL}(d, 2)$. Since $G_{\{A,B\}}$ is an index 3 subgroup of the stabiliser in $\mathrm{GL}(d, 2)$ of the 2-space $\langle v, w\rangle$, it contains a Sylow 2-subgroup of $\mathrm{GL}(d, 2)$. Thus H_0 contains a Sylow 2-subgroup of $\mathrm{GL}(d, 2)$ and it follows from a lemma of Tits (see for example [34, (1.6)]) that H_0 is a parabolic subgroup and hence is a subspace stabiliser. Now let $G = C_2^4 \rtimes A_7$. Since $G_{\{A,B\}} \cong S_4$ fixes a nonzero vector it is contained in a subgroup $\mathrm{PSL}(2, 7)$ of A_7 and hence by [10, p. 10], the elements of order 3 in $G_{\{A,B\}}$ are from the conjugacy class $3B$, that is, in the representation of A_7 on 7 points they are products of two 3-cycles. By [10, p. 10], A_7 has 5 conjugacy classes of maximal subgroups. The elements of order 3 in a maximal S_5 subgroup are from the conjugacy class $3A$ [10, p. 10], instead of $3B$ and so $H_0 \ncong S_5$. If $H_0 \cong A_6$ then $A_6 \cong \mathrm{PSp}(4, 2)'$ and contains two conjugacy classes of S_4 subgroups. One is the stabiliser of a vector and has orbit lengths 1, 6 and 8 on nonzero vectors and the other is the stabiliser of a totally isotropic 2-space with orbit sizes 3 and 12. Hence none of them stabilises the pair $\{v, w\}$ and so $H_0 \ncong A_6$. The remaining three conjugacy classes of maximal subgroups of A_7 are stabilisers of subspaces. Thus for both groups G, H_0 is a subspace stabiliser. The only proper, nontrivial subspaces fixed by $G_{\{A,B\}}$ are $\langle v + w\rangle$ and $\langle v, w\rangle$. If $H_0 = (G_0)_{\langle v,w\rangle}$ then H is the stabiliser of the class of planes parallel to $\langle v, w\rangle$ and so H is the stabiliser of the divisor containing $\{A, B\}$ of the decomposition in row 2 of Table 9. Similarly, if $H_0 = (G_0)_{\langle v+w\rangle}$ then H is the stabiliser of the class of lines parallel to $\langle v + w\rangle$ and so is the stabiliser of the divisor containing $\{A, B\}$ of the decomposition in row 4 of Table 9. □

8. Completing the case $k = 2$: $G \leqslant \mathrm{P\Gamma L}(2, q)$

In this section we determine all G-primitive decompositions of $J(q + 1, 2)$ where G is a 3-transitive subgroup of $\mathrm{P\Gamma L}(2, q)$ for $q = p^f \geqslant 4$ with p a prime. The group $\mathrm{PGL}(2, q)$ is the group of all fractional linear transformations

$$t_{a,b,c,d} : z \mapsto \frac{az + b}{cz + d}, \quad ad - bc \neq 0$$

of the projective line $X = \{\infty\} \cup \mathrm{GF}(q)$ with the conventions $1/0 = \infty$ and $(a\infty + b)/(c\infty + d) = a/c$. Note that $t_{a,b,c,d} = t_{a',b',c',d'}$ if and only if $(a, b, c, d) = \lambda(a', b', c', d')$ for some $\lambda \neq 0$. The group $\mathrm{PSL}(2, q)$ is then the set of all $t_{a,b,c,d}$ such that $ad - bc$ is a square in $\mathrm{GF}(q)$. The Frobenius map $\phi : z \mapsto z^p$ also acts on X and $\phi^{-1} t_{a,b,c,d} \phi = t_{a^p,b^p,c^p,d^p}$. Then $\mathrm{P\Gamma L}(2, q) = \langle \mathrm{PGL}(2, q), \phi\rangle$. Another interesting family of subgroups of $\mathrm{P\Gamma L}(2, q)$ occurs when p is odd and f is even. In this case we can define for each divisor s of $f/2$, the group $M(s, q) = \langle \mathrm{PSL}(2, q), \phi^s t_{\xi,0,0,1}\rangle$, where ξ is a primitive element of $\mathrm{GF}(q)$. Each $g \in \mathrm{PGL}(2, q) \setminus \mathrm{PSL}(2, q)$ can be written as $t_{\xi,0,0,1} h$ for some $h \in \mathrm{PSL}(2, q)$, and so $\phi^s g \in M(s, q)$. It was shown in [17, Theorem 2.1] that G is a 3-transitive subgroup of $\mathrm{P\Gamma L}(2, q)$ if and only if either G contains $\mathrm{PGL}(2, q)$, or $G = M(s, q)$ for some s.

We begin with the following construction.

Construction 8.1. (See [11].) Let $X = \{\infty\} \cup \mathrm{GF}(q)$ be the projective line, $H = \mathrm{PSL}(2, q)$ and $q \equiv 1 \pmod 4$. Then H is has two equal sized orbits on edges, namely $P_\square = \{\{\infty, 0\}, \{\infty, 1\}\}^H$, and $P_{\not\square} = \{\{\infty, 0\}, \{\infty, t\}\}^H$, with t not a square in $\mathrm{GF}(q)$. Thus the partition $\mathcal{P} = \{P_\square, P_{\not\square}\}$ is a G-primitive decomposition of $J(q + 1, 2)$ for any 3-transitive subgroup G of $\mathrm{P\Gamma L}(2, q)$. The divisors are complementary spanning graphs Θ of valency $q - 1$.

Proposition 8.2. *Let G be a 3-transitive subgroup of* $\mathrm{P\Gamma L}(2,q)$ *and let $\mathcal{P}$ be a G-primitive decomposition of $J(q+1,2)$ such that* $\mathrm{PSL}(2,q)$ *fixes a part. Then $q \equiv 1 \pmod 4$ and $\mathcal{P}$ is obtained from Construction* 8.1.

Proof. The graph $J(q+1,2)$ contains $\frac{q(q^2-1)}{2}$ edges. For q even, $|\mathrm{PSL}(2,q)| = q(q^2-1)$ and an edge stabiliser has order 2, so $\mathrm{PSL}(2,q)$ is transitive on edges. Thus q is odd and so $|\mathrm{PSL}(2,q)| = \frac{q(q^2-1)}{2}$. Whenever $(q-1)/2$ is odd, the stabiliser in $\mathrm{PSL}(2,q)$ of a point of X has odd order. Since the stabiliser of the edge $\{\{x,y\},\{x,z\}\}$ fixes x and interchanges y and z, it follows that no nontrivial element of $\mathrm{PSL}(2,q)$ fixes an edge and so $\mathrm{PSL}(2,q)$ is edge-transitive. Hence $(q-1)/2$ is even and $\mathrm{PSL}(2,q)$ has two equal length orbits on edges, giving the G-primitive decomposition of Construction 8.1 for any 3-transitive subgroup G of $\mathrm{P\Gamma L}(2,q)$. □

To classify all G-primitive decompositions with G a 3-transitive subgroup of $\mathrm{P\Gamma L}(2,q)$ we require knowledge of the maximal subgroups of all such G. First we note the following theorem.

Theorem 8.3. *(See [18, Corollary 1.2].) Let* $\mathrm{PGL}(2,q) \leqslant G \leqslant \mathrm{P\Gamma L}(2,q)$ *and suppose that H is a maximal subgroup of G not containing* $\mathrm{PSL}(2,q)$. *Then* $H \cap \mathrm{PGL}(2,q)$ *is maximal in* $\mathrm{PGL}(2,q)$.

Theorem 8.3 and Lemma 2.7 imply that we only need to find all $\mathrm{PGL}(2,q)$-primitive and all $M(s,q)$-primitive decompositions. We now state all maximal subgroups of these two groups. The first is well known and follows from Dickson's classification [15] of subgroups of $\mathrm{PSL}(2,q)$, see also [18].

Theorem 8.4. *Let $G = \mathrm{PGL}(2,q)$ with $q \geqslant 4$ a power of the prime p. Then the maximal subgroups of G are*:

(1) $[q] \rtimes C_{q-1}$.
(2) $D_{2(q-1)}$, $q \neq 5$.
(3) $D_{2(q+1)}$.
(4) S_4 *if* $q = p \equiv \pm 3 \pmod 8$.
(5) $\mathrm{PGL}(2,q_0)$ *where $q = q_0^r$ with $q_0 > 2$, r is a prime and r is odd if q odd.*
(6) $\mathrm{PSL}(2,q)$, *q odd.*

Theorem 8.5. *(See [18, Theorem 1.5].) Let $G = M(s,q)$ with $q = p^f \geqslant 3$ for p odd and f even, and s a divisor of $f/2$. Then the maximal subgroups of G which do not contain* $\mathrm{PSL}(2,q)$ *are*:

(1) *stabiliser of a point of the projective line,*
(2) $N_G(D_{q-1})$,
(3) $N_G(D_{q+1})$,
(4) $N_G(\mathrm{PSL}(2,q_0))$ *where $q = q_0^r$ with r an odd prime.*

We require the following knowledge about the stabiliser of an edge.

Lemma 8.6. *Let $e = \{\{\infty,0\},\{\infty,1\}\}$. Then*

(1) $\mathrm{PGL}(2,q)_e = \langle t_{-1,1,0,1}\rangle$,

(2) $\mathrm{P\Gamma L}(2,q)_e = \langle t_{-1,1,0,1}, \phi\rangle$ *of order* $2f$, *and*
(3) $M(s,q)_e = \langle t_{-1,1,0,1}, \phi^{2s}\rangle$ *of order* f/s.

Proof. Since $\mathrm{PGL}(2,q)$ is sharply 3-transitive, $\mathrm{PGL}(2,q)_e = \langle g\rangle$ where g fixes ∞ and interchanges 0 and 1. Thus $\mathrm{PGL}(2,q)_e$ is as in the lemma. Since ϕ fixes ∞, 0 and 1, the second claim follows. By [17, Corollary 2.2], $M(s,q)_{\infty,0,1} = \langle \phi^{2s}\rangle$ and since q is an even power of a prime we have $q \equiv 1 \pmod 4$. Thus $t_{-1,1,0,1} \in \mathrm{PSL}(2,q)$ and so $M(s,q)_e$ is as given by the lemma. □

Instead of finding all maximal subgroups H containing the stabiliser of a fixed edge $\{A,B\}$ we solve the equivalent problem of choosing a representative H from each conjugacy class of maximal subgroups and finding all edges whose edge stabiliser is contained in H. See Remark 2.5.

Construction 8.7. Let $X = \{\infty\} \cup \mathrm{GF}(q)$ be the projective line with q odd and let $H = \mathrm{P\Gamma L}(2,q)_\infty = \mathrm{A\Gamma L}(1,q)$. Let $e = \{\{0,1\},\{0,-1\}\}$. The stabiliser in $\mathrm{P\Gamma L}(2,q)$ of e is $\langle \phi, t_{-1,0,0,1}\rangle$, which is contained in H. Moreover H is a maximal subgroup of $\mathrm{P\Gamma L}(2,q)$. Thus by Lemma 2.4, letting

$$P = e^H = \bigl\{\bigl\{\{i,i+j\},\{i,i-j\}\bigr\} \bigm| i,j \in \mathrm{GF}(q),\ i \neq j\bigr\}$$

and $\mathcal{P} = P^{\mathrm{P\Gamma L}(2,q)}$, we obtain a $\mathrm{P\Gamma L}(2,q)$-primitive decomposition of $J(q+1,2)$. The divisors have valency 2 and hence are a union of cycles. Since $\mathrm{GF}(q)$ has characteristic p it follows that each cycle has length p and so the divisors are isomorphic to $\frac{q(q-1)}{2p}C_p$. For any 3-transitive group G with socle $\mathrm{PSL}(2,q)$, $H \cap G$ is maximal in G and so $\mathcal{P}$ is G-primitive by Lemma 2.7.

Proposition 8.8. *Let* $(J(q+1,2),\mathcal{P})$ *be a* G*-primitive decomposition with* G *a* 3*-transitive subgroup of* $\mathrm{P\Gamma L}(2,q)$ *such that, for* $P \in \mathcal{P}$, G_P *is the stabiliser of a point of the projective line. Then either* $\mathcal{P} = \mathcal{P}_\cap$ *with divisors* K_q *or* q *is a power of an odd prime* p *and* $\mathcal{P}$ *is obtained by Construction* 8.7.

Proof. Let $P \in \mathcal{P}$ and $\Gamma = J(q+1,2)$. Then without loss of generality we may suppose that $H = G_P$ is the stabiliser of the point ∞ of $X = \{\infty\} \cup \mathrm{GF}(q)$. We recall that G either contains $\mathrm{PGL}(2,q)$ or is $M(s,q)$ for some s. Thus H acts 2-transitively on $\mathrm{GF}(q)$ and so the orbits of H on $V\Gamma$ are $O_1 = \{\{\infty,x\} \mid x \in \mathrm{GF}(q)\}$ and $O_2 = \{\{x,y\} \mid x,y \in \mathrm{GF}(q)\}$. If $\{A,B\} \in P$ then H contains the stabiliser in G of $\{A,B\}$ and so either $\{A,B\} \subseteq O_1$ or $\{A,B\} \subseteq O_2$. Note that $P = \{A,B\}^H$.

Since H is 2-transitive on $\mathrm{GF}(q)$ it follows that H acts transitively on the set of arcs between vertices of O_1 and so H contains the stabiliser in G of every edge between vertices of O_1. Thus if $\{A,B\} \subseteq O_1$ then

$$\{A,B\}^H = \bigl\{\bigl\{\{\infty,x\},\{\infty,y\}\bigr\} \bigm| x,y \in \mathrm{GF}(q)\bigr\} \cong K_q.$$

Hence $\mathcal{P} = \mathcal{P}_\cap$.

Suppose now that $\{A,B\} \subseteq O_2$. We may suppose that $A = \{0,1\}$ and $B = \{0,b\}$ for some $b \in \mathrm{GF}(q)\backslash\{0,1\}$. Let $g = t_{0,b,1-b,b} \in \mathrm{PGL}(2,q)$. Then g maps $\infty \to 0 \to 1 \to b$ and so $G_{\{A,B\}} = G^g_{\{\{\infty,0\},\{\infty,1\}\}}$ (this is obvious if G contains $\mathrm{PGL}(2,q)$ and follows from the fact that $M(s,q) \lhd \mathrm{P\Gamma L}(2,q)$ for $G = M(s,q)$). By Lemma 8.6, $t^g_{-1,1,0,1} \in G_{\{A,B\}} \leqslant H = G_\infty$, and since g does not fix ∞ and the only fixed points of $t_{-1,1,0,1}$ are ∞ and 2^{-1} (only if q is odd), it follows that q is odd and $g : 2^{-1} \to \infty$. This implies that $b = -1$. Hence ϕ^g fixes ∞ and so by Lemma 8.6, $G_{\{\{0,1\},\{0,-1\}\}} \leqslant H$ in all cases. Hence $\mathcal{P}$ is the decomposition of Construction 8.7. □

8.1. D_{q-1} subgroups

Construction 8.9. Let $X = \{\infty\} \cup \mathrm{GF}(q)$ be the projective line where $q = p^f$ for some odd prime p and let ξ be a primitive element of $\mathrm{GF}(q)$. Then $\mathrm{P\Gamma L}(2,q)_{\{0,\infty\}} = \langle t_{\xi,0,0,1}, t_{0,1,1,0}, \phi\rangle \cong D_{2(q-1)} \rtimes C_f$.

(1) Let $H = \mathrm{P\Gamma L}(2,q)_{\{0,\infty\}}$ and $e = \{\{0,1\},\{0,-1\}\}$. Then $t_{-1,0,0,1} \in H$ interchanges the two vertices of e while ϕ fixes each of the vertices of e. Hence H contains the stabiliser in $\mathrm{P\Gamma L}(2,q)$ of e and H is a maximal subgroup of $\mathrm{P\Gamma L}(2,q)$ for $q \neq 5$. Thus by Lemma 2.4, letting
$$P = e^H = \bigl\{\bigl\{\{x,y\},\{x,-y\}\bigr\} \bigm| x \in \{0,\infty\},\ y \in \mathrm{GF}(q)\backslash\{0\}\bigr\}$$
and $\mathcal{P} = P^{\mathrm{P\Gamma L}(2,q)}$, we obtain a $\mathrm{P\Gamma L}(2,q)$-primitive decomposition of $J(q+1,2)$. The divisors are isomorphic to $(q-1)K_2$ since the stabiliser of the vertex $\{0,1\}$ in H is $\langle\phi\rangle$, which fixes $\{0,-1\}$. For any 3-transitive subgroup G of $\mathrm{P\Gamma L}(2,q)$, we have $H \cap G$ is maximal in G and so $\mathcal{P}$ is a G-primitive decomposition by Lemma 2.7.

(2) Let $i < \frac{q-1}{2}$ and l be an integer such that ϕ^l fixes the set $\{\xi^i, \xi^{-i}\}$. Let $G = \langle \mathrm{PGL}(2,q), \phi^l\rangle$ and $H = G_{\{\infty,0\}} = \langle t_{\xi,0,0,1}, t_{0,1,1,0}, \phi^l\rangle$. The automorphism of $\mathrm{PGL}(2,q)$ switching the vertices of the edge $e = \{\{1,\xi^i\},\{1,\xi^{-i}\}\}$ is $t_{0,1,1,0}$, while either ϕ^l or $t_{0,1,1,0}\phi^l$ fixes both vertices of e. Hence $G_e < H$ and H is a maximal subgroup of G for $q \neq 5$. Hence by Lemma 2.4, letting
$$P = e^H = \bigl\{\bigl\{\{x,\xi^i x\},\{x,\xi^{-i}x\}\bigr\} \bigm| x \in \mathrm{GF}(q)\backslash\{0\}\bigr\}$$
and $\mathcal{P} = P^G$, we obtain a G-primitive decomposition of $J(q+1,2)$. The divisors have valency 2 and hence are a union of cycles. These cycles have length the order of ξ^i, which is $\frac{q-1}{(q-1,i)}$. Thus each divisor is isomorphic to $(q-1,i)C_{\frac{q-1}{(q-1,i)}}$. In fact for any 3-transitive subgroup $\overline{G}$ of G, $H \cap \overline{G}$ is maximal in $\overline{G}$ and so $\mathcal{P}$ is a $\overline{G}$-primitive decomposition.

Proposition 8.10. *Let $(J(q+1,2),\mathcal{P})$ be a G-primitive decomposition such that* $\mathrm{PGL}(2,q) \leqslant G \leqslant \mathrm{P\Gamma L}(2,q)$ *and for $P \in \mathcal{P}$ we have $G_P = N_G(D_{2(q-1)})$. Then either $\mathcal{P} = \mathcal{P}_\ominus$, or q is odd and $\mathcal{P}$ is obtained by Construction* 8.9(1)*, or $\mathcal{P}$ is obtained by Construction* 8.9(2).

Proof. Let $P \in \mathcal{P}$. Since $G_P \cap \mathrm{PGL}(2,q)$ is a maximal subgroup of $\mathrm{PGL}(2,q)$, by Lemma 2.7, $\mathcal{P}$ is a $\mathrm{PGL}(2,q)$-primitive decomposition. Thus we may suppose that $G = \mathrm{PGL}(2,q)$ and $H = G_P = \langle t_{\xi,0,0,1}, t_{0,1,1,0}\rangle \cong D_{2(q-1)}$. The orbits of H on vertices are $\{\{0,\infty\}\}$,
$$O_0 = \bigl\{\{x,y\} \bigm| x \in \{0,\infty\},\ y \in \mathrm{GF}(q)\backslash\{0\}\bigr\}$$
and
$$O_i = \bigl\{\{x,\xi^i x\} \bigm| x \in \mathrm{GF}(q)\backslash\{0\}\bigr\}$$
for each $i \leqslant \frac{q-1}{2}$. Note that $|O_0| = 2(q-1)$. When q is even there are $q/2-1$ orbits O_i, each having length $q-1$. When q is odd there are $\frac{q-3}{2}$ of length $q-1$ and one, $O_{\frac{q-1}{2}}$, of length $\frac{q-1}{2}$.

If $\{A,B\} \in P$ then H contains the stabiliser in G of $\{A,B\}$ and so $\{A,B\}$ is contained in one of the orbits of H on vertices. Note that $P = \{A,B\}^H$.

Suppose first that $\{A,B\} \subseteq O_0$. Without loss, let $A = \{0,1\}$. Then the neighbours of A in O_0 are $\{\infty,1\}$ and $\{0,y\}$ such that $y \in \mathrm{GF}(q)\backslash\{0\}$. The only ones which can be interchanged with

A by an element of H are $\{\infty, 1\}$, by $t_{0,1,1,0}$ and $\{0, -1\}$, by $t_{-1,0,0,1}$, when q is odd. Thus the only edges between vertices of O_0 whose stabiliser in G is contained in H are those in the orbits $\{A, \{\infty, 1\}\}^H$ and $\{A, \{0, -1\}\}^H$. The first gives the matching $\{\{\{0, y\}, \{\infty, y\}\} \mid y \in \mathrm{GF}(q)\backslash\{0\}\}$ and hence the decomposition $\mathcal{P}_\ominus$ while the second gives the matching $\{\{\{x, y\}, \{x, -y\}\} \mid x \in \{0, \infty\},\ y \in \mathrm{GF}(q)\backslash\{0\}\}$ and hence Construction 8.9(1). Both matchings have $q - 1$ edges and the second only occurs for q odd. Note also that both orbits are preserved by $\mathrm{P\Gamma L}(2, q)_{\{0,\infty\}}$ and so both decompositions are also $\mathrm{P\Gamma L}(2, q)$-decompositions.

Note that when q is odd the orbit $O_{\frac{q-1}{2}}$ contains no edges. Thus suppose next that $\{A, B\} \subseteq O_i$ for $i < \frac{q-1}{2}$. Without loss of generality, let $A = \{1, \xi^i\}$. Then the neighbours of A in O_i are $\{1, \xi^{-i}\}$ and $\{\xi^i, \xi^{2i}\}$ and these are interchanged by $H_A = \langle t_{0,\xi^i,1,0}\rangle \cong C_2$. Hence H acts transitively on the set of edges between vertices of O_i. Moreover, $\langle t_{0,1,1,0}\rangle$ is the stabiliser H of the edge $\{\{1, \xi^i\}, \{1, \xi^{-i}\}\}$ and so H contains the stabiliser in G of an edge between two vertices of O_i. Thus $\mathcal{P}$ is obtained by Construction 8.9(2). Moreover, an overgroup $\overline{G} = \langle \mathrm{PGL}(2, q), \phi^l\rangle$ of $\mathrm{PGL}(2, q)$ in $\mathrm{P\Gamma L}(2, q)$ preserves $\mathcal{P}$ if and only if $\overline{G}_{\{0,\infty\}} = \langle H, \phi^l\rangle$ fixes O_i. Since ϕ^l fixes 1, it follows that ϕ^l fixes O_i if and only if ϕ^l fixes $\{\xi^i, \xi^{-1}\}$ and so $\overline{G}$ is as stated in Construction 8.9(2). $\square$

Construction 8.11. Let $G = M(s, q)$ and ξ be a primitive element of $\mathrm{GF}(q)$ with $q = p^f$ for some odd prime p and even integer f. Let i be an integer and assume that either

- $s = f/2$ and $(\xi^i)^{\langle\phi^s\rangle}$ has length 2 and does not contain ξ^{-i}, or
- $s = f/4$ and $(\xi^i)^{\langle\phi^s\rangle}$ has length 4 and does contain ξ^{-i}.

Let $H = G_{\{0,\infty\}} = \langle \mathrm{PSL}(2, q)_{\{0,\infty\}}, \phi^s t_{\xi,0,0,1}\rangle$ and note that $\mathrm{PSL}(2, q)_{\{0,\infty\}} = \langle t_{\xi^2,0,0,1}, t_{0,1,1,0}\rangle$.

(1) Suppose that i is even and let $e = \{\{1, \xi^i\}, \{1, \xi^{-i}\}\}$ and $P = e^H$. Then

$$P = \{\{\{x^2, x^2\xi^i\}, \{x^2, x^2\xi^{-i}\}\} \mid x \in \mathrm{GF}(q) \setminus \{0\}\} \\ \cup \{\{\{y, y\xi^{ip^s}\}, \{y, y\xi^{-ip^s}\}\} \mid y = \not\square\}.$$

Then P has valency 2 (as the two neighbours of $\{1, \xi^i\}$ are $\{1, \xi^{-i}\}$ and $\{\xi^i, \xi^{2i}\}$) and so is a union of cycles. Each cycle has length the order of ξ^i and so $P \cong (q - 1, i)C_{\frac{q-1}{(q-1,i)}}$.
Now $|\{1, \xi^i\}^H| = q - 1$ and by Lemma 8.6, $|G_e| = f/s$. Since $|H| = (q - 1)f/s$ it follows that $|H_e| = f/s$ and so $H_e = G_e$. Hence by Lemma 2.4 and the fact that H is maximal in G, letting $\mathcal{P} = P^G$ we get that $\mathcal{P}$ is a G-primitive decomposition.

(2) Suppose now that i is odd and let $e = \{\{1, \xi^i\}, \{1, \xi^{-i}\}\}$ and $P = e^H$. Then

$$P = \{\{\{x^2, x^2\xi^i\}, \{x^2, x^2\xi^{-i}\}\} \mid x \in \mathrm{GF}(q)\backslash\{0\}\} \\ \cup \{\{\{y, y\xi^{ip^s}\}, \{y, y\xi^{-ip^s}\}\} \mid y = \not\square\}.$$

Then $|P| = q - 1$ and so $|H_e| = f/s = |G_e|$, by Lemma 8.6. The only neighbour of $\{1, \xi^i\}$ in P is $\{1, \xi^{-i}\}$ and so $P = (q - 1)K_2$. By Lemma 2.4 and the fact that H is maximal in G, letting $\mathcal{P} = P^G$ we get that $\mathcal{P}$ is a G-primitive decomposition.

Proposition 8.12. *Let $(J(q + 1, 2), \mathcal{P})$ be a G-primitive decomposition with $G = M(s, q)$ for some s such that for $P \in \mathcal{P}$, $G_P = N_G(D_{q-1})$. Then either $\mathcal{P} = \mathcal{P}_\ominus$, or $\mathcal{P}$ arises from Construction* 8.9(1), 8.9(2) *or* 8.11.

Proof. A subgroup $N_G(D_{q-1})$ of G is a pair-stabiliser in G. Without loss of generality we may suppose that $H = G_{\{0,\infty\}} = \langle \mathrm{PSL}(2,q)_{\{0,\infty\}}, \phi^s t_{\xi,0,0,1}\rangle$. Note that $q \equiv 1 \pmod 4$ and so $\mathrm{PSL}(2,q)_{\{0,\infty\}} = \langle t_{\xi^2,0,0,1}, t_{0,1,1,0}\rangle$. Since G is 3-transitive it follows that

$$O_0 = \bigl\{\{x,y\} \bigm| x \in \{0,\infty\},\ y \in \mathrm{GF}(q)\backslash\{0\}\bigr\}$$

is an H-orbit on vertices and as in the proof of Lemma 8.10, if $\{A,B\} \subset O_0$ is an edge whose stabiliser in G is contained in H we obtain either $\mathcal{P} = \mathcal{P}_\ominus$ or $\mathcal{P}$ is obtained by Construction 8.9(1).

Now suppose $\{A,B\} \not\subset O_0$. Since H is transitive on $\mathrm{GF}(q)\backslash\{0\}$, we can assume that $A = \{1,\xi^i\}$ where $1 \leqslant i \leqslant q-2$ and that $A \cap B = \{1\}$, say $B = \{1,t\}$. We need to find the neighbours B of A such that $G_{\{A,B\}} \leqslant H$. Let $g \in \mathrm{PGL}(2,q)$ map $\{\{\infty,0\},\{\infty,1\}\}$ onto $\{A,B\}$. Then $G_{\{A,B\}} = \langle t_{-1,1,0,1}, \phi^{2s}\rangle^g$ by Lemma 8.6. Hence $t_{-1,1,0,1}$ and ϕ^{2s} must stabilise $\{0,\infty\}^{g^{-1}}$. Note that $\infty^g \neq \infty$ (since $\infty \notin A$) and $\infty^g \neq 0$ (since $0 \notin A$).

Since $B = \{1,t\}$, we can take $g = t_{a,\xi^i,a,1}$ where $a = \frac{\xi^i - t}{t-1}$, and then $\{0,\infty\}^{g^{-1}} = \{-\frac{\xi^i}{a}, -\frac{1}{a}\}$. Recall that $t_{-1,1,0,1}$ stabilises this set. Now $t_{-1,1,0,1}$ fixes only the points $\infty, 2^{-1}$, and if $\{0,\infty\}^{g^{-1}} = \{\infty, 2^{-1}\}$ we would have $\infty^g \in \{0,\infty\}$ which is not the case. Hence $t_{-1,1,0,1}$ interchanges $-\frac{\xi^i}{a}$ and $-\frac{1}{a}$. Thus $-\frac{\xi^i}{a} = 1 + \frac{1}{a}$, that is $a = -1 - \xi^i = \frac{\xi^i - t}{t-1}$, and so $t = \xi^{-i}$. For this value of t, $\{0,\infty\}^{g^{-1}} = \{\frac{\xi^i}{\xi^i+1}, \frac{1}{\xi^i+1}\}$ and this set is stabilised by $t_{-1,1,0,1}$ and ϕ^{2s}. The equality $\{\frac{\xi^i}{1+\xi^i}, \frac{1}{1+\xi^i}\}^{\phi^{2s}} = \{\frac{\xi^i}{1+\xi^i}, \frac{1}{1+\xi^i}\}$, is equivalent to either $\frac{\xi^{ip^{2s}}}{1+\xi^{ip^{2s}}} = \frac{\xi^i}{1+\xi^i}$ and $\frac{1}{1+\xi^{ip^{2s}}} = \frac{1}{1+\xi^i}$, or $\frac{\xi^{ip^{2s}}}{1+\xi^{ip^{2s}}} = \frac{1}{1+\xi^i}$ and $\frac{1}{1+\xi^{ip^{2s}}} = \frac{\xi^i}{1+\xi^i}$. In the first case $\xi^{ip^{2s}} = \xi^i$; in the second case $\xi^{ip^{2s}} = \xi^{-i}$. That means $O = (\xi^i)^{\langle\phi^s\rangle}$ has length 1, 2 or 4.

Set $e = \{A, \{1,\xi^{-1}\}\}$. If O has length 1, or O has length 2 and $(\xi^i)^{\phi^s} = \xi^{-i}$, then e^H yields a decomposition in Construction 8.9(2). If O has length 2 and $(\xi^i)^{\phi^s} \neq \xi^{-i}$, or O has length 4 and $\xi^{ip^{2s}} = \xi^{-i}$, then e^H yields a decomposition Construction 8.11(1) if i is even and in Construction 8.11(2) if i is odd. □

8.2. D_{q+1} subgroups

Before dealing with the case where $H \cap \mathrm{PSL}(2,q) = D_{q+1}$ we need a new model for the group action. Let $K = \mathrm{GF}(q^2)$ for $q = p^f$ with primitive element ξ, and let $F = \{0\} \cup \{(\xi^{q+1})^l \mid l = 0,1,\ldots,q-2\} \cong \mathrm{GF}(q)$. The element ξ acts on K by multiplication and induces an F-linear map. Moreover, under the induced action of F, K is a 2-dimensional vector space over F. The field automorphism φ of K of order $2f$ mapping each element of K to its pth power is F-semilinear, that is, φ preserves addition and for each $x \in K$, $\lambda \in F$, we have $(\lambda x)^\varphi = \lambda^p x^\varphi$. Then $\Gamma\mathrm{L}(2,q) = \langle \mathrm{GL}(2,q), \varphi\rangle$. Note that φ^f is an F-linear map so $\varphi^f \in \mathrm{GL}(2,q)$.

We can identify the projective line X on which $\mathrm{PGL}(2,q)$ acts with the elements of K modulo F, that is, $X = \{\xi^i F \mid i = 0,1,\ldots,q\}$. Then $\mathrm{P}\Gamma\mathrm{L}(2,q) = \langle \mathrm{PGL}(2,q), \varphi\rangle$. Multiplication by ξ induces the map $\hat{\xi}$ of order $q+1$ and $\langle\hat{\xi}\rangle$ is normalised by φ. Moreover, for each i, $(\xi^i F)^{\varphi^f} = \xi^{iq} F = \xi^{-i} F$ and so φ^f inverts $\hat{\xi}$. Hence $\langle \hat{\xi}, \varphi^f\rangle \cong D_{2(q+1)}$.

Construction 8.13. Let X be the projective line modelled as above. Let $1 \leqslant i < \frac{q+1}{2}$ and $e = \{\{1F, \xi^i F\}, \{1F, \xi^{-i} F\}\}$ and let s be a positive integer dividing f such that $\langle\varphi^s\rangle$ has $\{\xi^i F, \xi^{-i} F\}$ as an orbit on X. Let $G = \langle \mathrm{PGL}(2,q), \varphi^s\rangle$ and $H = \langle \hat{\xi}, \varphi^s\rangle \cong C_{q+1} \rtimes C_{2f/s}$. Now $\langle\varphi^s\rangle$ fixes e

and has order $2f/s$, which by Lemma 8.6 is the order of G_e. Hence $G_e < H$ and H is a maximal subgroup of G. Thus by Lemma 2.4, letting

$$P = e^H = \left\{\left\{\left\{xF, x\xi^i F\right\}, \left\{xF, x\xi^{-i} F\right\}\right\} \mid x \in \mathrm{GF}(q)\backslash\{0\}\right\}$$

and $\mathcal{P} = P^G$, we obtain a G-primitive decomposition of $J(q+1,2)$. The divisors have valency 2 and hence are unions of cycles. These cycles have length the order of $\xi^i F$, which is $\frac{q+1}{(q+1,i)}$. Thus each divisor is isomorphic to $(q+1,i)C_{\frac{q+1}{(q+1,i)}}$.

Proposition 8.14. *Let $(J(q+1,2),\mathcal{P})$ be a G-primitive decomposition such that $\mathrm{PGL}(2,q) \leqslant G \leqslant \mathrm{P\Gamma L}(2,q)$ and, for $P \in \mathcal{P}$, $G_P = N_G(D_{2(q+1)})$. Then $\mathcal{P}$ is obtained by Construction* 8.13.

Proof. Since $\mathrm{P\Gamma L}(2,q) = \langle \mathrm{PGL}(2,q), \varphi\rangle$ and $\varphi^f \in \mathrm{PGL}(2,q)$, we have $G = \langle \mathrm{PGL}(2,q), \varphi^s\rangle$ for some s dividing f. Let $L = \langle \hat{\xi}, \varphi^f\rangle \cong D_{2(q+1)}$. Then $N_G(L) = \langle \hat{\xi}, \varphi^s\rangle \cong C_{q+1} \rtimes C_{2f/s}$ and we may assume that $H = G_P = N_G(L)$. Let $e \in P$. Since H is transitive on X we may also assume that $e = \{\{1F, \xi^i F\}, \{1F, \xi^j F\}\}$ for some integers i and j. Since $H_{1F} = \langle \varphi^s\rangle$ and by Lemma 8.6, $|G_e| = 2f/s$, it follows that $G_e \leqslant H$ if and only if $\langle \varphi^s\rangle$ has $\{\xi^i F, \xi^j F\}$ as an orbit on X. Since $\varphi^f \in \langle \varphi^s\rangle$ and maps $\xi^i F$ to $\xi^{-i} F$ it follows that $j = -i$. Since $\xi^{-i} F = \xi^{q+1-i} F$ we may assume that $1 \leqslant i \leqslant (q+1)/2$. Moreover, if $i = (q+1)/2$ then q is odd and $\xi^{-(q+1)/2} F = \xi^{(q+1)/2} F$. Thus we may further assume that $1 \leqslant i < (q+1)/2$. Hence $\mathcal{P}$ arises from Construction 8.13. □

Next we need the following lemma about the normaliser in $M(s,q)$ of a subgroup D_{q+1} in $\mathrm{PSL}(2,q)$.

Lemma 8.15. *Suppose $q = p^f$ where f is even and p is an odd prime. Let $L = \langle \hat{\xi}, \varphi^f\rangle \cap \mathrm{PSL}(2,q)$ and $G = M(s,q)$ for some divisor s of $f/2$. Then*

(1) $L = \langle \hat{\xi}^2, \varphi^f\rangle \cong D_{q+1}$.
(2) *If $p \equiv 1 \pmod 4$ or s is even then $N_G(L) = \langle \hat{\xi}^2, \varphi^s\hat{\xi}\rangle$, and is transitive on the projective line.*
(3) *If $p \equiv 3 \pmod 4$ and s is odd then $N_G(L) = \langle \hat{\xi}^2, \varphi^s\rangle$, and has two equal sized orbits on the projective line.*

Proof. Now $\{1, \xi^{(q+1)/2}\}$ is a basis for K over F. Define $\phi : K \to K$ such that, for all $\lambda_1, \lambda_2 \in F$, $(\lambda_1 + \lambda_2\xi^{(q+1)/2})^\phi = \lambda_1^p + \lambda_2^p\xi^{(q+1)/2}$. Then $\Gamma\mathrm{L}(2,q) = \langle \mathrm{GL}(2,q), \phi\rangle$. Since also $\Gamma\mathrm{L}(2,q) = \langle \mathrm{GL}(2,q), \varphi\rangle$, we must have $\varphi = \phi g$ for some $g \in \mathrm{GL}(2,q)$. Since φ and ϕ fix 1, so does g. Moreover, ϕ fixes $\xi^{(q+1)/2}$ while $(\xi^{(q+1)/2})^\varphi = \xi^{p(q+1)/2} = \xi^{\frac{(p-1)(q+1)}{2}}\xi^{\frac{q+1}{2}}$. Note that $\xi^{\frac{(p-1)(q+1)}{2}} \in F$ and so $\xi^{(q+1)/2}$ is an eigenvector for g. Thus with respect to the basis $\{1, \xi^{(q+1)/2}\}$, the element g is represented by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & \xi^{\frac{(p-1)(q+1)}{2}} \end{pmatrix},$$

and $\det(g) = \xi^{\frac{(p-1)(q+1)}{2}}$ is a square in $\mathrm{GF}(q)$ if and only if $p \equiv 1 \pmod 4$. Furthermore, φ^f is represented by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Recall that an element of $\mathrm{GL}(2,q)$ induces an element of $\mathrm{PSL}(2,q)$ if and only if its determinant is a $\mathrm{GF}(q)$-square. Since $q \equiv 1 \pmod 4$ it follows that $\varphi^f \in \mathrm{PSL}(2,q)$. Now $\langle \hat{\xi}^2 \rangle \cong C_{(q+1)/2}$ and $\hat{\xi}^2 \in \mathrm{PSL}(2,q)$, and since φ^f inverts $\hat{\xi}$ it also inverts $\hat{\xi}^2$. Hence L is as in part (1) of the lemma. Moreover, L has two orbits on the projective line X, these being $\{1F, \xi^2 F, \ldots, \xi^{q-1}F\}$ and $\{\xi F, \xi^3 F, \ldots, \xi^q F\}$.

Now $\varphi = \phi g$ and $g \in \mathrm{PSL}(2,q)$ if and only if $p \equiv 1 \pmod 4$. By definition it follows that $G = M(s,q) = \langle \mathrm{PSL}(2,q), \phi^s t\rangle$ for any $t \in \mathrm{PGL}(2,q) \setminus \mathrm{PSL}(2,q)$. Suppose first that $p \equiv 1 \pmod 4$. Then $\varphi = \phi g$ with $g \in \mathrm{PSL}(2,q)$ and so $G = \langle \mathrm{PSL}(2,q), \varphi^s \hat{\xi}\rangle$. When $p \equiv 3 \pmod 4$ we have $\varphi = \phi g$ with $g \in \mathrm{PGL}(2,q) \setminus \mathrm{PSL}(2,q)$. Thus for odd s we have $G = \langle \mathrm{PSL}(2,q), \varphi^s\rangle$ while for even s we have $G = \langle \mathrm{PSL}(2,q), \varphi^s \hat{\xi}\rangle$. Now $(\varphi^f)^{\varphi^s \hat{\xi}} = (\varphi^f)^{\hat{\xi}} = \varphi^f \hat{\xi}^{-p^s+1} \in L$. Hence for $p \equiv 1 \pmod 4$ or s even we have $N_G(L) = \langle \hat{\xi}^2, \varphi^s \hat{\xi}\rangle$. Since $\varphi^s \hat{\xi}$ interchanges the two L-orbits on X, $N_G(L)$ is transitive on X and so we have proved part (2). For $p \equiv 3 \pmod 4$ and s odd we have $N_G(L) = \langle \hat{\xi}^2, \varphi^s\rangle$. Since φ^s fixes each L-orbit it follows that $N_G(L)$ has two orbits and the proof is complete. □

Construction 8.16. Let $q = p^f$ where p is odd and f even and let $G = M(s,q)$ for some divisor s of $f/2$. Suppose that either $p \equiv 1 \pmod 4$ or s is even. Let $1 \leqslant i < (q+1)/2$ such that $\langle \varphi^{2s}\rangle$ has $\{\xi^i F, \xi^{-i} F\}$ as an orbit on X. Let $H = \langle \hat{\xi}^2, \varphi^s \hat{\xi}\rangle$ and $e = \{\{1F, \xi^i F\}, \{1F, \xi^{-i} F\}\}$. Now $\langle \varphi^{2s}\rangle$ fixes e, lies in G, and has order f/s. Since this is the same order as G_e (Lemma 8.6) it follows that $G_e < H$. Hence by Lemma 2.4, letting $P = e^H$ and $\mathcal{P} = P^G$ we obtain a G-primitive decomposition.

(1) Suppose first that i is even. Then $H_{\{1F, \xi^i F\}} = \langle \varphi^f \hat{\xi}^i, \varphi^{4s}\rangle$ whose orbit containing $\{1F, \xi^{-i}F\}$ is $\{\{1F, \xi^{-i}F\}, \{\xi^i F, \xi^{2i} F\}\}$. Thus P has valency 2 and so is a union of cycles of length the order of $\hat{\xi}^i$, that is, $P \cong (q+1, i) C_{\frac{q+1}{(q+1,i)}}$.
(2) Suppose now that i is odd. An element of H mapping $1F$ to $\xi^i F$ is of the form $h = \varphi^{st} \hat{\xi}^i$ with t odd. Since $\langle \varphi^{2s}\rangle$ has $\{\xi^i F, \xi^{-i} F\}$ as an orbit on X, we have that h maps $\xi^i F$ onto $\xi^{i(1+p^s)} F$ or onto $\xi^{i(1-p^s)} F$, according as $t \equiv 1$ or $3 \pmod 4$ respectively. Hence, for h to map $\xi^i F$ onto $1F$, we need $q+1$ to divide $i(1+p^s)$ or $i(1-p^s)$ respectively. Since $p^{2s}-1$ divides $p^f - 1 = q - 1$, it follows that $\gcd(q+1, p^s+1) = 2$ and $\gcd(q+1, p^s-1) = 2$, and so $\frac{q+1}{2}$ must divide i, which is not possible since $1 \leqslant i < \frac{q+1}{2}$. Thus $(\xi^i F)^h \neq 1F$. Hence $H_{\{1F, \xi^i F\}} = H_{1F, \xi^i F} = \langle \varphi^{4s}\rangle$, which also fixes $\xi^{-i} F$ and hence fixes e. Thus P is a matching with $q+1$ edges.

Construction 8.17. Let $p \equiv 3 \pmod 4$ and let $G = M(s,q)$ for $q = p^f$ and s an odd divisor of $f/2$. Let $1 \leqslant i < (q+1)/2$ such that $\langle \varphi^{2s}\rangle$ has $\{\xi^i F, \xi^{-i} F\}$ as an orbit on X. Let $H = \langle \hat{\xi}^2, \varphi^s\rangle$ and $e = \{\{1F, \xi^i F\}, \{1F, \xi^{-i} F\}\}$. Now $\langle \varphi^{2s}\rangle$ fixes e, lies in G and has order f/s. Since this is the same order as G_e (Lemma 8.6) it follows that $G_e < H$ and so by Lemma 2.4, letting $P = e^H$ and $\mathcal{P} = P^G$, we obtain a G-primitive decomposition.

(1) Suppose first that i is even. Then $H_{\{1F, \xi^i F\}} = \langle \varphi^f \hat{\xi}^i, \varphi^{4s}\rangle$ and the H-orbit containing $\{1F, \xi^{-i} F\}$ has length 2. Thus P is a union of cycles of length the order of $\hat{\xi}^i$, so $P \cong (q+1, i) C_{\frac{q+1}{(q+1,i)}}$.
(2) If i is odd then $1F$ and $\xi^i F$ lie in different H-orbits and so $H_{\{1F, \xi^i F\}} = H_{1F, \xi^i F} = \langle \varphi^{4s}\rangle$ which also fixes $\xi^{-i} F$ and hence fixes e. Thus P is a matching with $q+1$ edges.

Construction 8.18. Let $p \equiv 3 \pmod 4$ and let $G = M(s,q)$ for $q = p^f$ and s an odd divisor of $f/2$. Let $1 \leqslant i < \frac{q+1}{2}$ such that $\langle \hat{\xi}^{-1}\varphi^{2s}\hat{\xi}\rangle$ has $\{\xi^{i+1}F, \xi^{-i+1}F\}$ as an orbit on X. Let $H = \langle \hat{\xi}^2, \varphi^s\rangle$ and $e = \{\{\xi F, \xi^{i+1}F\}, \{\xi F, \xi^{-i+1}F\}\}$. Now $\langle \hat{\xi}^{-1}\varphi^{2s}\hat{\xi}\rangle \leqslant H$, fixes e, and has the same order as G_e. Thus $G_e < H$ and so by Lemma 2.4, letting $P = e^H$ and $\mathcal{P} = P^G$, we obtain a G-primitive decomposition.

(1) Suppose first that i is odd. Then ξF and $\xi^{i+1}F$ lie in different H-orbits. Hence $H_{\{\xi F, \xi^{i+1}F\}} = H_{\xi F, \xi^{i+1}F} = \langle \hat{\xi}^{-1}\varphi^{4s}\hat{\xi}\rangle$ which also fixes $\xi^{-i+1}F$ and so P is a matching with $q+1$ edges.
(2) If i is even then $\varphi^f\hat{\xi}^{i+2} \in H$ interchanges ξF and $\xi^{i+1}F$, and so $H_{\{\xi F, \xi^{i+1}F\}} = \langle \hat{\xi}^{-1}\varphi^{4s}\hat{\xi}, \varphi^f\hat{\xi}^{i+2}\rangle$, whose orbit containing $\{\xi F, \xi^{-i+1}F\}$ has size 2. Hence P is a union of cycles of length the order of $\hat{\xi}^i$. Thus $P = (q+1, i)C_{\frac{q+1}{(q+1,i)}}$.

Proposition 8.19. *Let $\mathcal{P}$ be an $M(s,q)$-primitive decomposition of $J(q+1,2)$ with divisor stabiliser $N_{M(s,q)}(D_{q+1})$. Then $\mathcal{P}$ can be obtained from Construction* 8.16*,* 8.17 *or* 8.18*.*

Proof. Let $G = M(s,q)$ and suppose first that $q = p^f$ where $p \equiv 1 \pmod 4$ or s is even. We may assume that $H = \langle \hat{\xi}^2, \varphi^s\hat{\xi}\rangle$ by Lemma 8.15. Let $e \in P \in \mathcal{P}$. By Lemma 8.15 again, H is transitive on X and so we can assume that $e = \{\{1F, \xi^iF\}, \{1F, \xi^jF\}\}$ for some i and j. Now $H_{1F} = \langle \varphi^{2s}\rangle$, which has order f/s. By Lemma 8.6, this is the same order as G_e. Hence $G_e < H$ if and only if $H_{1F} = G_e$, which holds if and only if $\{\xi^iF, \xi^jF\}$ is an orbit of $\langle \varphi^{2s}\rangle$. Since $\varphi^f \in \langle \varphi^{2s}\rangle$ and maps ξ^iF to $\xi^{-i}F$ it follows that $j = -i$ and we may assume as before that $1 \leqslant i < (q+1)/2$. Thus $\mathcal{P}$ comes from Construction 8.16.

Suppose now that $p \equiv 3 \pmod 4$ and s is odd. Then by Lemma 8.15, we may assume that $H = \langle \hat{\xi}^2, \varphi^s\rangle$. Let $e \in P \in \mathcal{P}$. By Lemma 8.15, H has 2 orbits on X and so we may assume that $e = \{\{1F, \xi^iF\}, \{1F, \xi^jF\}\}$ or $\{\{\xi F, \xi^{i+1}F\}, \{\xi F, \xi^{j+1}F\}\}$. Suppose that e is the first edge. Now $H_{1F} = \langle \varphi^s\rangle$ which has order $2f/s$ while G_e has order f/s by Lemma 8.6. Since H_{1F} has a unique subgroup of order f/s it follows that $G_e < H$ if and only if $G_e = \langle \varphi^{2s}\rangle$, that is, if and only if $\langle \varphi^{2s}\rangle$ has $\{\xi^iF, \xi^jF\}$ as an orbit on X. Since $\varphi^f \in \langle \varphi^{2s}\rangle$ we have $j = -i$ and may assume $1 \leqslant i < (q+1)/2$. It follows that $\mathcal{P}$ is as constructed in Construction 8.17. If on the other hand $e = \{\{\xi F, \xi^{i+1}F\}, \{\xi F, \xi^{j+1}F\}\}$, then $H_{\xi F} = \langle \hat{\xi}^{-1}\varphi^s\hat{\xi}\rangle$ which has order $2f/s$. Its only index two subgroup is $\langle \hat{\xi}^{-1}\varphi^{2s}\hat{\xi}\rangle$ and so by order arguments again this must have $\{\xi^{i+1}F, \xi^{j+1}F\}$ as an orbit. Since $\hat{\xi}^{-1}\varphi^f\hat{\xi} \in \langle \hat{\xi}^{-1}\varphi^{2s}\hat{\xi}\rangle$ and maps $\xi^{i+1}F$ to $\xi^{-i+1}F$ it follows that $j = -i$. Once again we have $1 \leqslant i < \frac{q+1}{2}$. Hence $\mathcal{P}$ is as given by Construction 8.18. □

8.3. S_4-subgroups

First we have the following lemma on the orbit lengths of a subgroup S_4 of $\mathrm{PGL}(2,q)$ which we have adapted from [8].

Lemma 8.20. *(See [8, Lemma 10].) Let $q = p \equiv \pm 3 \pmod 8$, $q > 3$, $G = \mathrm{PGL}(2,q)$ acting on the projective line X, and H a subgroup of G isomorphic to S_4. Then H has the following orbits of length less than* 24 *on X.*

(1) *If $q \equiv 5 \pmod{24}$, then H has one orbit of length* 6*.*
(2) *If $q \equiv 11 \pmod{24}$, then H has one orbit of length* 12*.*

(3) *If* $q \equiv 13 \pmod{24}$, *then* H *has one orbit of length* 6 *and one of length* 8.
(4) *If* $q \equiv 19 \pmod{24}$, *then* H *has one orbit of length* 8 *and one of length* 12.

Construction 8.21. Let $X = \{\infty\} \cup \mathrm{GF}(q)$ be the projective line.

(1) Let $q \equiv \pm 3 \pmod 8$ be a prime ($q > 3$) and $H = S_4$. Choose $x, y_1, y_2 \in X$ such that $(|x^H|, |y_1|^H) = (6, 8), (6, 24), (12, 8)$ or $(12, 24)$, and there exists in H_x an element switching y_1 and y_2. Let $P = \{\{x, y_1\}, \{x, y_2\}\}^H$ and $\mathcal{P} = P^{\mathrm{PGL}(2,q)}$. Then by Lemma 2.4, $(J(q+1, 2), \mathcal{P})$ is a $\mathrm{PGL}(2,q)$-primitive decomposition. Since $|\{x, y_1\}|^H = 24$, the stabiliser in H of $\{x, y_1\}$ is trivial. Hence the divisors are isomorphic to $12K_2$.
(2) Let $q \equiv 5 \pmod 8$ be a prime and $H = S_4$. Let $P = \{\{x, y_1\}, \{x, y_2\}\}^H$ where x, y_1, y_2 all lie in an H-orbit of length 6 and there exists in H_x an element switching y_1 and y_2. By Lemma 8.20, there is a unique orbit of O_6 of length 6. The group H acts imprimitively on O_6 with blocks of size 2, and $H_x \cong C_4$ contains an element interchanging y_1 and y_2 if and only if $\{y_1, y_2\}$ is a block not containing x. Moreover, $P \cong 3C_4$. Let $\mathcal{P} = P^{\mathrm{PGL}(2,q)}$. Then by Lemma 2.4 $(J(q+1, 2), \mathcal{P})$, is a $\mathrm{PGL}(2,q)$-primitive decomposition.
(3) Let $q \equiv 3 \pmod 8$ be a prime and $H = S_4$. Let $P = \{\{x, y_1\}, \{x, y_2\}\}^H$ where x, y_1, y_2 all lie in an H-orbit of length 12 and there exists in H_x an element switching y_1 and y_2. By Lemma 8.20, there is a unique orbit O_{12} of length 12. We can see this action as S_4 acting on ordered pairs, denoted by $[a, b]$. Then for $x = [1, 2] \in O_{12}$, H_x is the transposition $(3, 4)$ in S_4. It fixes one remaining point of O_{12}, namely $[2, 1]$ and interchanges the 5 pairs $\{[2, 3], [2, 4]\}$, $\{[3, 1], [4, 1]\}$, $\{[1, 3], [1, 4]\}$, $\{[3, 2], [4, 2]\}$, and $\{[3, 4], [4, 3]\}$. If we take $\{y_1, y_2\}$ as in the first two cases, then the stabiliser in H of $\{x, y_1\}$ is trivial and so we get a matching $12K_2$ in each case. In the last three cases, the stabiliser in H of $\{x, y_1\}$ has order 2, and we get unions of cycles. It is easy to see that in the third and fourth case, we get $4C_3$, while in the last case we get $3C_4$. Let $\mathcal{P} = P^{\mathrm{PGL}(2,q)}$. Then by Lemma 2.4, $(J(q+1, 2), \mathcal{P})$ is a $\mathrm{PGL}(2,q)$-primitive decomposition.

Proposition 8.22. *Let* $(J(q+1, 2), \mathcal{P})$ *be a* G*-primitive decomposition with* $G = \mathrm{PGL}(2,q)$ *for* $q = p \equiv \pm 3 \pmod 8$ *with* $q \geqslant 5$ *and given* $P \in \mathcal{P}$ *we have* $G_P \cong S_4$. *Then* P *is obtained by Construction* 8.21(1), (2) *or* (3).

Proof. Let $P \in \mathcal{P}$ and $H = G_P \cong S_4$. If $\{x, y\} \subseteq X$ with x and y in different H-orbits of length 24 then $|\{x, y\}^H| = 24$ and that orbit contains no edges of $J(q+1, 2)$. Thus if x and y come from different H-orbits O_1 and O_2 respectively, we may assume by Lemma 8.20, that $|O_1| < |O_2|$ and so $\{x, y\}^H$ has length $\mathrm{lcm}(|O_1|, |O_2|)$ and contains edges. Moreover, H contains the stabiliser in G of such an edge $\{\{x, y_1\}, \{x, y_2\}\}$ if and only if H_x contains an element interchanging y_1 and y_2. If x is in an orbit of size 8 then $|H_x| = 3$ and so no such element exists, and if x is in an orbit of size 24 then $|H_x| = 1$ and so no such element exists. Thus the possibilities for $(|O_1|, |O_2|)$ are $(6, 8)$, $(6, 24)$, $(8, 12)$ or $(12, 24)$. In the first two cases x must be in the orbit of length 6 and in the last two cases x must be in the orbit of length 12. Thus we get the decomposition of Construction 8.21(1).

Suppose now $e = \{\{x, y_1\}, \{x, y_2\}\}$ is an edge such that $\{x, y_1, y_2\}$ lie in the same H-orbit O_i. Then H contains G_e if and only if H_x interchanges y_1 and y_2. Thus $|H_x|$ is even and so $|O_i| \neq 8, 24$. If $q \equiv 5 \pmod 8$ and O_i is the unique orbit of size 6 then we obtain the decomposition in Construction 8.21(2). If $q \equiv 3 \pmod 8$ and O_i is the unique orbit of size 12 then we obtain the decompositions in Construction 8.21(3). □

8.4. Subfield subgroups

Suppose now that $q = q_0^r$. Then $S = \{\infty\} \cup \mathrm{GF}(q_0)$ is a subset of the projective line $X = \{\infty\} \cup \mathrm{GF}(q)$ which is an orbit of the subgroup $\mathrm{P\Gamma L}(2, q_0)$ of $\mathrm{P\Gamma L}(2, q)$. Notice that ϕ fixes the set S. Moreover, by [9, I, Example 3.23], if $\mathcal{B} = S^{\mathrm{PGL}(2,q)}$ then $(X, \mathcal{B})$ is a $S(3, q_0 + 1, q + 1)$ Steiner system. Since ϕ fixes S and $\mathrm{P\Gamma L}(2, q) = \langle \mathrm{PGL}(2, q), \phi\rangle$ it follows that $\mathcal{B} = S^{\mathrm{P\Gamma L}(2,q)}$. Thus by Lemma 2.11, we can construct a $\mathrm{P\Gamma L}(2, q)$-transitive decomposition of $J(q + 1, 2)$ with divisors isomorphic to $J(q_0 + 1, 2)$. The stabiliser of a divisor is $\mathrm{P\Gamma L}(2, q_0)$. Moreover, this decomposition is G-transitive for any 3-transitive subgroup G of $\mathrm{P\Gamma L}(2, q)$. For further constructions we need the orbits of $\mathrm{PGL}(2, q_0)$ on $\mathrm{GF}(q) \setminus \mathrm{GF}(q_0)$.

Lemma 8.23. *(See [8, Lemma 14].) Let $q = q_0^r$ for some prime r and let $H = \{t_{a,b,c,d} \mid a, b, c, d \in \mathrm{GF}(q_0), ad - bc \neq 0\}$. If r is odd then H acts semiregularly on $\mathrm{GF}(q) \setminus \mathrm{GF}(q_0)$, while if $r = 2$ then H is transitive on $\mathrm{GF}(q) \setminus \mathrm{GF}(q_0)$.*

Construction 8.24. Let $X = \{\infty\} \cup \mathrm{GF}(q)$ be the projective line. Let $q = q_0^r$, where $q_0 > 2$, r is a prime and r is odd if q is odd. Let $e = \{\{\infty, w_1\}, \{\infty, w_2\}\}$ such that $w_1, w_2 \in \mathrm{GF}(q) \setminus \mathrm{GF}(q_0)$ but $w_1 + w_2 \in \mathrm{GF}(q_0)$. Let l be a positive integer such that ϕ^l fixes $\{w_1, w_2\}$. Then let $G = \langle \mathrm{PGL}(2, q), \phi^l\rangle$ and $H = \langle \mathrm{PGL}(2, q_0), \phi^l\rangle$. Let $P = e^H$ and $\mathcal{P} = P^G$. Then by Lemma 8.6, $G_e = \langle t_{-1,w_1+w_2,0,1}, \phi^l\rangle$ which is in H. Therefore by Lemma 2.4, $(J(q + 1, 2), \mathcal{P})$ is a G-primitive decomposition. The stabiliser $H_{\{\infty,w_1\}}$ fixes ∞ and w_1 as they are in different H-orbits. We claim that $\mathrm{PGL}(2, q_0)_{\infty,w_1} = 1$. Indeed, an element in that subgroup must be of the form $t_{a,b,0,1}$ with $a, b \in \mathrm{GF}(q_0)$, whose only fixed point is $\frac{b}{1-a} \in \mathrm{GF}(q_0)$ if it is not the identity. Hence there is a unique element of $\mathrm{PGL}(2, q_0)_\infty$ interchanging w_1 and w_2, this being $t_{-1,w_1+w_2,0,1}$. Then as ϕ^l fixes $\{w_1, w_2\}$ and ∞, it follows that H_{∞,w_1} fixes w_2. Hence P is isomorphic to $\frac{q_0(q_0^2-1)}{2} K_2$.

Proposition 8.25. *Let $(J(q + 1, 2), \mathcal{P})$ be a G-primitive decomposition such that $\mathrm{PGL}(2, q) \leqslant G \leqslant \mathrm{P\Gamma L}(2, q)$ and for $P \in \mathcal{P}$, $G_P \cong N_G(\mathrm{PGL}(2, q_0))$ where $q = q_0^r$, $q_0 > 2$, r is a prime and r is odd if q is odd. Then $\mathcal{P}$ is obtained by Construction* 2.10 *or Construction* 8.24.

Proof. By Theorem 8.3, $\mathcal{P}$ is also a $\mathrm{PGL}(2, q)$-primitive decomposition so we may suppose that $G = \mathrm{PGL}(2, q)$ and $H = G_P = \{t_{a,b,c,d} \mid a, b, c, d \in \mathrm{GF}(q_0), ad - bc \neq 0\}$. We have already seen that H has an orbit $\{\infty\} \cup \mathrm{GF}(q_0)$ of length $q_0 + 1$ on X. Moreover, by Lemma 8.23, when r is odd, H has $q_0^{r-3} + q_0^{r-5} + \cdots + q_0^2 + 1$ other orbits, all of length $q_0(q_0^2 - 1)$, while when $r = 2$, H is transitive on $\mathrm{GF}(q) \setminus \mathrm{GF}(q_0)$.

Suppose that H contains the stabiliser in G of the edge $e = \{\{v, w_1\}, \{v, w_2\}\}$. Then H_v contains the unique nontrivial element interchanging w_1 and w_2 (see Lemma 8.6). Now v must lie in the unique H-orbit of length $q_0 + 1$. For, if r is odd and v lies in an H-orbit of length $q_0(q_0^2 - 1)$ then $H_v = 1$, while if $r = 2$ and v lies in $\mathrm{GF}(q) \setminus \mathrm{GF}(q_0)$, then $|H_v| = q_0 + 1$ which is odd. Without loss of generality we may suppose that $v = \infty$.

Then $G_e = \langle t_{-1,w_1+w_2,0,1}\rangle$, so $G_e \leqslant H$ if and only if $w_1 + w_2 \in \mathrm{GF}(q_0)$. If w_1 and w_2 lie in the orbit of length $q_0 + 1$, that is, are in $\mathrm{GF}(q_0)$ then we obtain the decomposition from Construction 2.10, which is in fact preserved by $\mathrm{P\Gamma L}(2, q)$. If $w_1 \notin \mathrm{GF}(q_0)$ and $w_2 = a - w_1$ with $a \in \mathrm{GF}(q_0)$, then we get a decomposition obtained from Construction 8.24. □

For a primitive element μ of $\mathrm{GF}(q_0)$, $t_{\mu,0,0,1} \in \mathrm{PGL}(2, q) \setminus \mathrm{PSL}(2, q)$. Thus $\phi^s t_{\mu,0,0,1} \in M(s, q)$ and normalises $\mathrm{PSL}(2, q_0)$. Hence $N_{M(s,q)}(\mathrm{PSL}(2, q_0)) = \langle \mathrm{PSL}(2, q_0), \phi^s t_{\mu,0,0,1}\rangle$.

We will need the following lemma.

Lemma 8.26. *Let $G = M(s,q)$ with $q = q_0^r = p^f$ for some odd primes r and p, and even integer f, and let $H = \langle \mathrm{PSL}(2,q_0), \phi^s t_{\mu,0,0,1}\rangle$ where μ is a primitive element of* $\mathrm{GF}(q_0)$. *Let $e = \{\{\infty, w_1\}, \{\infty, w_2\}\}$.*

(1) *Then $G_e \leqslant H$ if and only if both $w_1 + w_2$ and $(w_2 - w_1)^{p^{2s}-1}$ lie in* $\mathrm{GF}(q_0)$.
(2) *There exist $w_1, w_2 \notin \mathrm{GF}(q_0)$ such that $w_1 + w_2$ and $(w_2 - w_1)^{p^{2s}-1}$ lie in $\mathrm{GF}(q_0)$ if and only if* $\gcd(\frac{q-1}{q_0-1}, p^{2s}-1) \neq 1$.

Proof. (1) By Lemma 8.6, $G_e = \langle t_{-1,w_1+w_2,0,1}, (\phi^{2s})^g\rangle$ where $g = t_{w_2-w_1,w_1,0,1}$. Since f is even and $q = q_0^r$ with r odd, q_0 is an even power of p and hence -1 is a square in $\mathrm{GF}(q_0)$. Thus $t_{-1,w_1+w_2,0,1} \in H$ if and only if $w_1 + w_2 \in \mathrm{GF}(q_0)$. Moreover,

$$\begin{aligned}
g^{-1}\phi^{2s}g &= t_{1,-w_1,0,w_2-w_1}\phi^{2s}t_{w_2-w_1,w_1,0,1}\\
&= \phi^{2s}t_{1,-w_1^{p^{2s}},0,(w_2-w_1)^{p^{2s}}}t_{w_2-w_1,w_1,0,1}\\
&= \phi^{2s}t_{w_2-w_1,-(w_2-w_1)w_1^{p^{2s}}+w_1(w_2-w_1)^{p^{2s}},0,(w_2-w_1)^{p^{2s}}}\\
&= \phi^{2s}t_{1,w_1(w_2-w_1)^{p^{2s}-1}-w_1^{p^{2s}},0,(w_2-w_1)^{p^{2s}-1}}.
\end{aligned}$$

Let $h = t_{1,w_1(w_2-w_1)^{p^{2s}-1}-w_1^{p^{2s}},0,(w_2-w_1)^{p^{2s}-1}}$. As $\phi^{2s} \in H$, it follows that $(\phi^{2s})^g \in H$ if and only if $h \in \mathrm{PSL}(2,q_0)$. Now if $h \in \mathrm{PSL}(2,q_0)$ then $(w_2 - w_1)^{p^{2s}-1} \in \mathrm{GF}(q_0)$. Thus if $G_e \leqslant H$ then both $w_1 + w_2$ and $(w_2 - w_1)^{p^{2s}-1}$ lie in $\mathrm{GF}(q_0)$. Conversely, suppose that $w_1 + w_2 = a \in \mathrm{GF}(q_0)$ and $w_2 - w_1 = u$ with $u^{p^{2s}-1} = b \in \mathrm{GF}(q_0)$. Then writing $\frac{1}{2}$ for $2^{-1} \in \mathrm{GF}(p)$ and noting that $2^{p^{2s}} = 2$, $w_1(w_2-w_1)^{p^{2s}-1} - w_1^{p^{2s}} = \frac{a-u}{2}b - \frac{a^{p^{2s}}-u^{p^{2s}}}{2^{p^{2s}}} = \frac{ab-a^{p^{2s}}}{2} \in \mathrm{GF}(q_0)$. Thus $h \in \mathrm{PGL}(2,q_0)$, and since $p^{2s}-1$ is even $h \in \mathrm{PSL}(2,q_0)$.

(2) Let ξ be a primitive element of $\mathrm{GF}(q)$. Then $\mathrm{GF}(q_0) = \{0\} \cup \{\xi^{i\frac{q-1}{q_0-1}} \mid i = 1,\ldots,q_0-1\}$ and we can choose $\mu = \xi^{\frac{q-1}{q_0-1}}$. For $w_2 - w_1 = \xi^j \in \mathrm{GF}(q) \setminus \{0\}$, if $(w_2-w_1)^{p^{2s}-1}$ lies in $\mathrm{GF}(q_0)$, that means that $\xi^{j(p^{2s}-1)} = \xi^{i\frac{q-1}{q_0-1}}$ for some integer i. If $\gcd(\frac{q-1}{q_0-1}, p^{2s}-1) = 1$, we must have j a multiple of $(q-1)/(q_0-1)$, and so $w_2 - w_1 \in \mathrm{GF}(q_0)$. If we also have $w_1 + w_2 \in \mathrm{GF}(q_0)$, then this implies that $w_1, w_2 \in \mathrm{GF}(q_0)$. Hence if $w_1, w_2 \notin \mathrm{GF}(q_0)$ such that $w_1 + w_2$ and $(w_2-w_1)^{p^{2s}-1}$ lie in $\mathrm{GF}(q_0)$ then $\gcd(\frac{q-1}{q_0-1}, p^{2s}-1) \neq 1$. Conversely, suppose $\gcd(\frac{q-1}{q_0-1}, p^{2s}-1) = d \neq 1$ and choose $j = (q-1)/d(q_0-1)$. Then take $w_2 = \xi^j/2$ and $w_1 = -\xi^j/2$. We obviously have $w_1 + w_2 \in \mathrm{GF}(q_0)$ and $w_1, w_2 \notin \mathrm{GF}(q_0)$. Moreover $(w_2-w_1)^{p^{2s}-1} = \xi^{\frac{p^{2s}-1}{d}\frac{q-1}{q_0-1}} \in \mathrm{GF}(q_0)$. $\square$

Construction 8.27. Let $G = M(s,q)$ and let $X = \{\infty\} \cup \mathrm{GF}(q)$ be the projective line. Let $q = q_0^r = p^f$ for some odd primes r and p, and f an even integer, and let $H = \langle \mathrm{PSL}(2,q_0), \phi^s t_{\mu,0,0,1}\rangle$ where μ is a primitive element of $\mathrm{GF}(q_0)$. Assume $\gcd(\frac{q-1}{q_0-1}, p^{2s}-1) \neq 1$, so that by Lemma 8.26, there exist $w_1, w_2 \notin \mathrm{GF}(q_0)$ such that $w_1 + w_2, (w_2-w_1)^{p^{2s}-1} \in \mathrm{GF}(q_0)$. Let $e = \{\{\infty, w_1\}, \{\infty, w_2\}\}$. By Lemma 8.6, $G_e = \langle t_{-1,w_1+w_2,0,1}, (\phi^{2s})^g\rangle$, where $g =$

$t_{w_2-w_1,w_1,0,1}$, and by Lemma 8.26, $G_e \leqslant H$. Thus letting $P = e^H$ and $\mathcal{P} = P^G$, $(J(q+1,2), \mathcal{P})$ is a G-primitive decomposition by Lemma 2.4.

We claim that the divisors of $\mathcal{P}$ are either matchings or unions of cycles. Since $\mathrm{PSL}(2,q_0)_\infty \lhd H_\infty = \langle \mathrm{PSL}(2,q_0)_\infty, \phi^s t_{\mu,0,0,1}\rangle$, H_∞ (of order $\frac{q_0(q_0-1)}{2}\frac{f}{s}$) acts on the set of $\mathrm{PSL}(2,q_0)_\infty$-orbits. Now $t_{-1,w_1+w_2,0,1} \in \mathrm{PSL}(2,q_0)_\infty$ interchanges w_1 and w_2, and hence w_1, w_2 lie in the same $\mathrm{PSL}(2,q_0)_\infty$-orbit, θ say. By Lemma 8.23, $\mathrm{PGL}(2,q_0)$ acts semiregularly on $\mathrm{GF}(q)\setminus \mathrm{GF}(q_0)$ and hence $|\theta| = |\mathrm{PSL}(2,q)_\infty| = \frac{q_0(q_0-1)}{2}$. Note that $H_{\{\infty,w_1\}} = H_{\infty,w_1}$ and $H_{\{\infty,w_2\}} = H_{\infty,w_2}$. Also $H_{\{\infty,w_1\},\{\infty,w_2\}} = \langle (\phi^{2s})^g\rangle$ has order $\frac{f}{2s}$. Notice that $(\phi^s t_{\mu,0,0,1})^2 = \phi^{2s} t_{\mu p^s+1,0,0,1}$. Hence $\langle \mathrm{PSL}(2,q_0)_\infty, (\phi^{2s})^g\rangle$ has index 2 in H_∞ and fixes θ. Therefore H_∞ either fixes θ or switches it with another $\mathrm{PSL}(2,q_0)_\infty$-orbit θ'. In the first case, $H_{\{\infty,w_1\}}$ has order $\frac{f}{s}$, while $H_{\{\infty,w_1\},\{\infty,w_2\}}$ has order $\frac{f}{2s}$, hence the divisor has valency 2 and is a union of cycles. In the second case, $H_{\{\infty,w_1\}}$ and $H_{\{\infty,w_1\},\{\infty,w_2\}}$ both have order $\frac{f}{2s}$, and so the divisor is a matching $\frac{q_0(q_o^2-1)}{2}K_2$.

Remark 8.28. We have not determined the length of the cycles occurring in the first case of Construction 8.27. This case happens if and only if there exists $w \in \mathrm{GF}(q) \setminus \mathrm{GF}(q_0)$ such that $w^{\phi^s t_{\mu,0,0,1}} = w^{p^s}\mu \in \{a^2 w + b \mid a,b \in \mathrm{GF}(q_0)\}$. We have not been able to find any instances where this condition holds.

Proposition 8.29. *Let $(J(q+1,2),\mathcal{P})$ be a G-primitive decomposition with $G = M(s,q)$ and for $P \in \mathcal{P}$ we have that $G_P = N_G(\mathrm{PSL}(2,q_0))$ where $q = q_0^r$ for some odd prime r. Then $\mathcal{P}$ is obtained by Construction* 2.10 *or Construction* 8.27.

Proof. Let $q = p^f$ with p a prime and f an even integer. As seen in the discussion before Lemma 8.26, $H := G_P = \langle \mathrm{PSL}(2,q_0), \phi^s t_{\mu,0,0,1}\rangle$ where μ is a primitive element of $\mathrm{GF}(q_0)$. Let $X = \{\infty\} \cup \mathrm{GF}(q)$. Then one orbit of H on X is $\{\infty\} \cup \mathrm{GF}(q_0)$. Since H is maximal in G, H is exactly the stabiliser in G of $\{\infty\} \cup \mathrm{GF}(q_0)$.

Suppose that H contains G_e for some edge $e = \{\{v,w_1\},\{v,w_2\}\}$.

Then by Lemma 8.6, H contains an element of $\mathrm{PSL}(2,q)$, and hence of $\mathrm{PSL}(2,q_0)$, which fixes v and interchanges w_1 and w_2. Since, by Lemma 8.23, $\mathrm{PSL}(2,q_0)$ acts semiregularly on $\mathrm{GF}(q) \setminus \mathrm{GF}(q_0)$, it follows that $v \in \{\infty\} \cup \mathrm{GF}(q_0)$. Without loss of generality we may suppose that $v = \infty$. By Lemma 8.26, this means that both $w_1 + w_2$ and $(w_2 - w_1)^{p^{2s}-1}$ lie in $\mathrm{GF}(q_0)$. This is of course satisfied if $w_1, w_2 \in \mathrm{GF}(q_0)$, and then we get Construction 2.10 using $\mathcal{B} = (\{\infty\} \cup \mathrm{GF}(q_0))^{\mathrm{P\Gamma L}(2,q)}$, as G is transitive on $\mathcal{B}$. Now assume $w_1, w_2 \notin \mathrm{GF}(q_0)$. Then by Lemma 8.26, $\gcd(\frac{q-1}{q_0-1}, p^{2s}-1) \neq 1$. Moreover, $P = e^H$ and $\mathcal{P} = P^G$ are as obtained in Construction 8.27. □

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